LANCASTER UNIVERSITY

2000 EXAMINATIONS

Part I

PHYSICS - Paper PS1.2

- *Candidates should attempt all those sections identified with the modules for which they are registered.*
- *Candidates who attended PHYS113, PHYS114 and/or PHYS115 attempt sections A, B and/or C.*
- *Candidates who attended PHYS113a, PHYS114a and/or PHYS115a attempt sections D, E and/or F.*
- *The time allocated is 60 minutes per section.*
- *An indication of mark weighting (30 marks per section) is given by the numbers in square brackets following each part.*
- *In each section attempted, candidates should answer question 1 (10 marks) and either question 2 or question 3 (20 marks).*
- *Use a separate answer book for each section.*

PHYS110

Section A: Module 113 - Complex Numbers and Calculus
For candidates who attended PHYS113 (NOT PHYS113a)

A1. (a) Find the modulus and the complex conjugate of $z = 9 - 12i$. [2]

(b) Rationalise the function $w = \dfrac{3+2i}{9-12i}$ into the form $x + iy$. [2]

(c) Convert the complex number $z = 9 - 12i$ into the polar form $r(\cos\theta + i\sin\theta)$ and plot this on an Argand diagram. [3]

(d) For the function $y = 4x^3 + 3x$, find the value of $\dfrac{dy}{dx}$ at $x = 0.5$. [3]

please turn over

A2. (a) What is the square root of the complex number $u = 0.90\, e^{i\theta}$ where $\theta = 2\pi/5$. [2]

Convert both u and $\sqrt{u}$ into Cartesian form. [4]

For $n = 0, 1, 2, 3, 4$, and 5, plot u^n on an Argand diagram. [4]

Regarding these points as marking positions on a smooth path, sketch the continuation of this path as $n \to \infty$. [2]

(b) The graph supplied shows the function $y(x) = x^2 + x^3$. By drawing tangents to the curve plotted, find the rate of change of $y(x)$ with respect to x at the points where $x = 1.0$ and $x = -0.5$. [4]

(Insert your name on the graph and attach this to your answer book.)

Find an expression for $\frac{dy}{dx}$ and use this to discuss the accuracy of the graphical method for evaluating a rate of change. [4]

A3. (a) Differentiate the following functions with respect to x:

(i) $y(x) = 7x^5$.

(ii) $y(x) = 7\sin(5x)$.

(iii) $y(x) = 7x^5 \sin(5x)$.

(iv) $y(x) = \dfrac{7x^5}{\cos(5x)}$.

(v) $y(x) = 7\log_{10}(5x)$.

(vi) $y(x) = 7x^5 \log_{10}(5x)$. [12]

(b) If $z = (\cos\theta + i\sin\theta)$, use Demoivre's theorem to show that

$$\cos(4\theta) = \frac{1}{2}\left(z^4 + \frac{1}{z^4}\right). \quad [2]$$

Hence or otherwise show that $\cos(4\theta) = 8\cos^4\theta - 8\cos^2\theta + 1$. [6]

Section B: Module 114 - Differentiation
For candidates who attended PHYS114 (NOT PHYS114a)

B1. The pressure P depends on the volume V through the equation

$$\left(P + \frac{a}{V^2}\right) V = RT,$$

where a, T and R are positive constants in this case.

(a) Find an expression for $P(V)$. [1]

(b) Find the conditions on V for which $P(V) = 0$ and $P(V) > 0$. [2]

(c) Differentiate P with respect to V. [3]

(d) Find the value of V for which the maximum value of P occurs. [2]

(e) What is the maximum value of P? [2]

B2. (a) Find the first derivatives of the following functions of x:

(i) $x\,e^{x^2}$,

(ii) $x\log_e(2x)$,

(iii) $\sin^2 x$. [6]

(b) Find $\frac{dy}{dx}$ for the implicit function

$$x^2 + 5xy + y^2 = 0. \quad [4]$$

(c) Find the minima, maxima and points of inflection of the function

$$f(x) = |x|\,e^{-x^2}.$$

Sketch the function. [10]

B3. (a) Compute the first and second derivatives of the following function

$$f(x) = \frac{e^x}{x+1}. \quad [4]$$

(b) Write down the general formula for the Taylor expansion of a function around $x = a$ and hence compute the Taylor expansion of $f(x)$ given in part (a) around $x = 0$ up to and including terms in x^2. [11]

(c) Compute the value of the Taylor expansion obtained in part (b) at $x = 0$ and compare it to the values of $f(x)$ there. Repeat this for $x = 1$ and comment on how the accuracy of the expansion could be improved. [5]

Section C: Module 115 - Integration
For candidates who attended PHYS115 (NOT PHYS115a)

C1. Evaluate the following definite integrals

(a)

$$\int_0^{\pi/2} \sin^2 x \cos x \, dx. \quad [3]$$

(b)

$$\int_{-2}^{-1} \frac{x}{\sqrt{x+2}} \, dx \quad [3]$$

(c)

$$\int_0^{\pi/2} x \cos x \, dx. \quad [4]$$

please turn over

C2. (a) Without performing any integrations, state whether each of the following statements are true or false and explain your answers:

(i)
$$\int_{-\pi/2}^{\pi/2} \cos^3 x \, dx = 0$$ [2]

(ii)
$$\int_{-\pi}^{\pi} \sin^2 x \, dx = 2\int_{0}^{\pi} \sin^2 x \, dx$$ [2]

(iii)
$$\int_{-1}^{1} x^2 \sin x \, dx = 0$$ [2]

(iv)
$$\int_{0}^{1} x^3 e^x \, dx < 0$$ [2]

(b) The velocity $v = \frac{dx}{dt}$ of an object performing damped simple harmonic motion is given by

$$v = Ae^{-\gamma t} \sin \omega t$$

where A, γ and ω are constants.

(i) Using integration (by parts or otherwise) show that the position of the object is given by

$$x = \frac{-A\gamma e^{-\gamma t}}{\gamma^2 + \omega^2}\left(\sin \omega t + \frac{\omega}{\gamma} \cos \omega t\right) + c$$

where c is a constant. [10]

(ii) If the object starts from position $x = 0$ at time $t = 0$, determine the constant c. [2]

C3. (a) Calculate the area enclosed by the graphs of $y = x^2$ and $y = x$. [6]

(b) A solid bar with circular cross-section lies along the x-axis from $x = 0$ to $x = L$. The bar has a variable radius given by

$$r(x) = R\sqrt{1 + \frac{x}{L}}$$

where R is a constant.

(i) Show that the total volume of the bar is $\frac{3}{2}\pi R^2 L$. [6]

(ii) If the density of the bar as a function of position x is given by

$$\rho = \rho_0 \exp\left(1 + \frac{x}{L}\right)$$

where ρ_0 is a constant, find the total mass of the bar and hence its average density. [8]

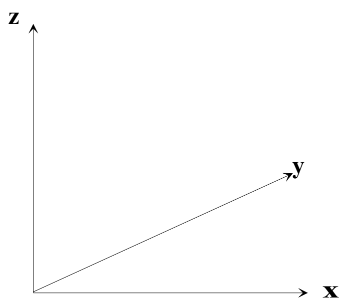
PHYS110a

Section D: Module 113a - Vectors
For candidates who attended PHYS113a (NOT PHYS113)

D1. (a) Give three examples of physical quantities which are scalars. [3]

(b) Give three examples of physical quantities which are vectors. [3]

$\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors parallel to the x, y and z axes of the Cartesian coordinate system shown in the figure below.



Evaluate the following

(c) $|\boldsymbol{k}|$. [1]

(d) $\boldsymbol{j} \times \boldsymbol{j}$. [1]

(e) $\boldsymbol{j} \cdot \boldsymbol{k}$. [1]

(f) $\boldsymbol{j} \times \boldsymbol{i}$. [1]

D2. When a body undergoes a displacement $\boldsymbol{S}$ whilst experiencing a force $\boldsymbol{F}$, the work done is $W = \boldsymbol{F} \cdot \boldsymbol{S}$.

(a) If $\boldsymbol{F}$ and $\boldsymbol{S}$ are non-zero, under what condition is $W = 0$. [3]

When $\boldsymbol{S} = (1, 2, 3)\,\mathrm{m}$ and $\boldsymbol{F} = (3, 1, 1)\,\mathrm{N}$

(b) find the work done, [3]

(c) compute $|\boldsymbol{F}|$ and $|\boldsymbol{S}|$, [6]

(d) find the angle θ between $\boldsymbol{F}$ and $\boldsymbol{S}$. [8]

please turn over

D3. When a particle with charge q and velocity $\mathbf{v}$ moves in a magnetic field $\boldsymbol{B}$, it experiences a force $\boldsymbol{F}= q\, \boldsymbol{v} \times \boldsymbol{B}$.

(a) Show on a sketch the direction of $\boldsymbol{F}$ in relation to $\boldsymbol{v}$ and $\boldsymbol{B}$. [3]

(b) For $q, |\boldsymbol{v}|,\ |\boldsymbol{B}|$ non-zero, discuss the condition for which $|\boldsymbol{F}| = 0$. [3]

(c) For the specific case $\boldsymbol{v} = (1, 2, 3)$ ms^{-1}, $\boldsymbol{B} = (4, 5, 6)$ T, and $q = 1$ C, compute:

(i) $\boldsymbol{F}$, [8]

(ii) $|\boldsymbol{F}|$, [3]

(iii) $\boldsymbol{F} \cdot \boldsymbol{B}$. [3]

Section E: Module 114a - Differentiation
For candidates who attended PHYS114a (NOT PHYS114)

E1. (a) Sketch graphs showing the acceleration, speed and position as a function of time for a body moving with uniform acceleration. State your assumptions for the conditions at $t = 0$. [3]

(b) The surface area of a sphere is $A = 4\pi r^2$. Find an expression for $\frac{dA}{dr}$. [1]

(c) For the function $y = 4x^3 + 3x$, find the value of $\frac{dy}{dx}$ at $x = 0.5$. [2]

(d) Differentiate the function $y = \log_e(5x^2 + 3x)$ with respect to x. [2]

(e) Find an expression giving $\frac{d^2y}{dt^2}$ for the function $y = y_0 \cos(\omega t + \phi)$. [2]

E2. (a) Differentiate the following functions with respect to x:

(i) $y(x) = 7x^5$.

(ii) $y(x) = 7 \sin(5x)$.

(iii) $y(x) = 7x^5 \sin(5x)$.

(iv) $y(x) = \dfrac{7x^5}{\cos(5x)}$.

(v) $y(x) = 7 \log_{10}(5x)$.

(vi) $y(x) = 7x^5 \log_{10}(5x)$. [12]

(b) The graph supplied shows the function $y(x) = x^2 + x^3$. By drawing tangents to the curve plotted, find the rate of change of $y(x)$ with respect to x at the points where $x = 1.0$ and $x = -0.5$. [4]

(Insert your name on the graph and attach this to your answer book.)

Find an expression for $\frac{dy}{dx}$ and use this to discuss the accuracy of the graphical method for evaluating a rate of change. [4]

E3. For each of the following functions

(a) $y = \cos(x^3)$,

(b) $y = \cos^3(x)$,

(c) $y = \log_e(\sin x)$,

(d) $y = \sin(\log_e x)$.

(i) Find $\frac{dy}{dx}$. [8]

(ii) Find $\frac{d^2y}{dx^2}$. [8]

(iii) Define the domain of x for which the function is valid. [4]

Section F: Module 115a - Further Calculus
For candidates who attended PHYS115a (NOT PHYS115)

F1. Find the minima, maxima and points of inflection of the function

$$f(x) = 6x^4 - 4x^3.$$ [10]

F2. Consider the function

$$f(x) = x\,e^{-x^2}.$$

(a) Discuss whether the function is even or odd. [2]

(b) Where does $f(x)$ cross the x-axis? [2]

(c) What are the asymptotes of $f(x)$ at large $|x|$? [2]

(d) Compute $\frac{df}{dx}$ and $\frac{d^2f}{dx^2}$. [5]

(e) Find the minima, maxima and points of inflection of $f(x)$. [6]

(f) Sketch the function. [3]

please turn over

F3. (a) Evaluate the following integrals

(i)

$$\int \cos 2x \, dx$$

(ii)

$$\int \frac{dx}{x^{3/2}}$$

(iii)

$$\int_1^2 \frac{dx}{x}$$

[10]

(b) Explain what is meant by an odd function and an even function. Without performing the actual integration, evaluate the integral

$$\int_{-1}^{1} e^{-x^4+x^2+1} \sin 2x \, dx.$$

[5]

(c) Find the relationship between the parameters m, k and ω in order for the function $y(t) = e^{\omega t}$ to be a solution of the differential equation

$$m\frac{d^2y(t)}{dt^2} = ky(t).$$

[5]