King's College London

UNIVERSITY OF LONDON

M.Sci. EXAMINATION

CP/4477 Electrons in Solids

Summer 2002

Time allowed: THREE Hours

Candidates must answer THREE questions.
No credit will be given for answering further questions.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

Avogadro Number $N_\mathrm{A} = 6.022 \times 10^{23}$ mol^{-1}

Boltzmann constant $k = 1.381 \times 10^{-23}$ J K^{-1}

Planck constant $h = 6.626 \times 10^{-34}$ J s

Charge of a proton $e = 1.602 \times 10^{-19}$ C

Rest mass of an electron $m = 9.109 \times 10^{-31}$ kg

Bohr magneton $\mu_\mathrm{B} = 9.274 \times 10^{-24}$ J T^{-1}

Permeability of free space $\mu_0 = 4\pi \times 10^{-7}$ H m^{-1}

Permittivity of free space $\epsilon_0 = 8.854 \times 10^{-12}$ F m^{-1}

GaAs has a direct energy gap of 1.43 eV, a relative permittivity of $\epsilon = 10.9$, and a density of valence electrons of 1.76×10^{29} m^{-3}. Its electron, heavy hole and light hole effective masses are 0.072, 0.6 and 0.15 times the free electron mass.

An electron, trapped in a *one-dimensional*, infinitely deep, square potential well of width a has energies $E = \hbar^2 k_x^2/2m$ measured from the bottom of the well. Here $k_x = \pi n_x/a$ and n_x is a positive integer. The corresponding wavefunctions are $\psi(x) = \sqrt{2/a}\sin(k_x x)$.

In the free-electron theory for a metal, the density of electron states at energy E is $g(E) = C\sqrt{E}$, where $C = 8\sqrt{2}\pi m^{3/2} V/h^3$, and V is the volume of the metal.

TURN OVER WHEN INSTRUCTED

Answer THREE questions

1) Describe a procedure for making GaAs quantum wires embedded in AlGaAs.

[3 marks]

A quantum wire has a square cross-section of width $a = 8$ nm, and has its major axis along the z direction. It is excited to create a low density of holes and electrons trapped in the wire. Using the information given at the head of the examination paper, estimate the lowest energy of the luminescence that is emitted when an electron and a hole recombine.

[4 marks]

Luminescence propagating along the x axis of a quantum wire is detected with its electric vector polarised perpendicular to the y axis. Show that the luminescence occurs only for an electron and a hole with identical quantum numbers n_x.

[5 marks]

A sample is made with GaAs quantum wires embedded in AlGaAs that is lightly doped with donors. Describe how this sample could be used to measure the offset potential of the conduction bands of GaAs and AlGaAs.

[3 marks]

Show that the density of electron states for a quantum wire of length 2 mm is $g(E) = 1.09 \times 10^{15} E^{-1/2}$, where the energy E, measured from each discrete quantum level, is in Joules.

[5 marks]

2) Gold (Au) has a free-electron density of $n = 5.9 \times 10^{28}$ m^{-3}. Show that, in the free-electron model, the magnitude of the wavevector at the Fermi surface of Au is $k_{\mathrm{F}} = 1.20 \times 10^{10}$ m^{-1}. [Useful data are at the head of the paper.]

[5 marks]

By equating the kinetic energy of a free electron to its cyclotron energy in a magnetic field, or by any other method, show that the period of de Haas van Alphen oscillations is $\Delta(1/B) = (2\pi)^2 e/hS$ where S is the area in k-space that is enclosed by the orbit.

[4 marks]

De Haas van Alphen oscillations are observed in Au with a period of $\Delta(1/B) = 2 \times 10^{-5}$ T^{-1}. Show that these oscillations correspond closely to an orbit at k_{F}.

[3 marks]

Sketch (a) the variation of the electron energy with wavevector in the free-electron model, and (b) the modification of that variation near the zone boundaries when a weak periodic potential is used.

[2 marks]

Au crystallises in the face-centred cubic structure. A long de Haas van Alphen period of $\Delta(1/B) = 6 \times 10^{-4}$ T^{-1} is observed when the magnetic field is along the $\langle 111 \rangle$ direction. Calculate the enclosed area of the orbit in k-space, and suggest an origin for the signal. Explain why the signal is detected for the field along $\langle 111 \rangle$.

[6 marks]

3) Answer all parts.

a) Sketch the energy distribution of occupied electron states at 0 K in the free-electron model, and show the modification at non-zero temperature. Using the sketch, demonstrate that the electrons make a contribution $C_e = aT$ to the heat capacity of a metal at temperature T. Estimate the value of a (in Joule mol^{-1} K^{-2}) for a monovalent metal whose Fermi energy is $E_F = 2.12$ eV.

[7 marks]

b) How could the energy quantum $\hbar\omega_p$ of a plasmon be measured for a thin film of GaAs? By considering a spatial displacement of all the valence electrons within a semiconductor, estimate $\hbar\omega_p$ for GaAs, using the data at the head of the paper.

State, giving your reasons, whether it is necessary to consider plasma oscillations in free-electron theory.

[6 marks]

c) The tight-binding approximation applied to a body-centred cubic crystal, in which only nearest neighbours interact, gives electronic energy levels of

$$E = A + 8B\cos(k_x a/2)\cos(k_y a/2)\cos(k_z a/2)$$

where a is the edge length of the unit cube.

Show that surfaces with constant energy are spherical when $\mathbf{k} \to 0$.

What is the effective mass component m_{zz} in terms of B at the point $k_x = k_y = 0$, $k_z = \pi/a$?

What is the mean energy of the electronic states? If each lattice site contributes two electrons to the metal, and if $B > 0$, what is the Fermi energy of the metal in terms of A and B?

[7 marks]

4) Answer all parts.

a) A hypothetical metal has n magnetic ions per unit volume. The ions each have non-interacting localised moments with allowed magnetic quantum numbers $m_{\rm J} = \pm\frac{1}{2}$, spin g-factor $g = 2$ and hence magnetic moments $\mu = m_{\rm J} g \mu_{\rm B} = \pm\mu_{\rm B}$. Show that the magnetic susceptibility (Curie susceptibility) of the localised moments is given by:

$$\chi = \frac{n\mu_0\mu_B}{B}\tanh\left(\frac{\mu_B B}{kT}\right).$$

[3 marks]

b) Derive an expression for the magnetic field that has to be applied to a sample of this metal such that a fraction η of the ions are in the lower energy state.

[4 marks]

Hence calculate the magnetic field required to give an average magnetic moment per ion of $0.8\mu_{\rm B}$, if the sample is at a temperature $T = 4.2$ K.

[3 marks]

c) If the magnetic moments now interact, the effective field acting on each ion may be written as $B = B_0 + \lambda M$, where B_0 is the external applied field, M is the magnetisation per unit volume, and λ is a molecular field constant. Show that the metal becomes ferromagnetic in zero applied field below a temperature

$$T_{\rm C} = n\lambda\mu_{\rm B}^2/k.$$

[4 marks]

The magnetisation per unit volume at temperature T, in the ferromagnetic state, is written as $M(T)$, and $\Delta M(T) = M(0) - M(T)$ describes the difference between $M(T)$ at a finite temperature and at $T = 0$. By considering a suitable expansion of $\tanh\xi$ for large ξ, show that at temperatures well below $T_{\rm C}$, $\Delta M(T)$ may be approximated by:

$$\Delta M(T)/M(0) = 2\exp\left(-2T_{\rm C}/T\right).$$

[4 marks]

Discuss briefly how well this result from molecular field theory agrees with typical experimental data, and compare the result from an alternative model.

[2 marks]

5) Write down an equation for the rate of change with time ($d\mathbf{J}_s/dt$) of the current density $\mathbf{J}_s$ that would occur if a superconductor had an electric field $\mathbf{E}$ inside it. Given the Maxwell equations

$$\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt} \quad \text{and} \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}_s,$$

and the general equation

$$\nabla \times \nabla \times \mathbf{B} = \nabla(\nabla \bullet \mathbf{B}) - \nabla^2 \mathbf{B},$$

show that the penetration depth λ of a magnetic field into the superconductor is given, in a simple theory, by

$$\lambda^2 = \frac{m}{N_s e^2 \mu_0}$$

where N_s is the concentration of superconducting electrons in the material.

What is the penetration depth in pure metallic tin (Sn), for which $N_s = 2.4 \times 10^{28}$ m^{-3}?

[6 marks]

If the coherence length in Sn is 230 nm, state, giving the reason for your answer, whether you expect Sn to be a type I or a type II superconductor.

[3 marks]

Describe how

a) the penetration depth λ could be measured for Sn, and

b) a coil of Cu wire plated with Sn may be used to demonstrate the quantisation of magnetic flux.

[8 marks]

The critical temperature of Sn is 3.7 K. Estimate the shortest wavelength of electromagnetic radiation that may be transmitted through a sample of superconducting Sn.

[3 marks]

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