King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP3270 Chaos in Physical Systems

Summer 2003

Time allowed: THREE Hours

Candidates must answer SIX parts of SECTION A, and TWO questions from SECTION B.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

SECTION A – Answer SIX parts of this section

1.1) Find the fixed points of the flow

$$\frac{dx}{dt} = \sin 2x.$$

and classify their stability.

[7 marks]

1.2) Convert the following equation into the standard form of a set of coupled first order differential equations:

$$\frac{d^3x}{dt^3} = -x^3.$$

[7 marks]

1.3) Use linear stability analysis to study the dynamical behaviour of the one-dimensional system

$$\frac{dx}{dt} = ax - bx^5$$

(a, b being real constants) for $a < 0$ and $b > 0$.

[7 marks]

1.4) Describe briefly the type of fluid dynamical system that is described by the Lorenz model.

[7 marks]

1.5) State the Poincaré-Bendixson theorem.

[7 marks]

1.6) Show that the mapping

$$x_{n+1} = x_n + y_{n+1}$$
$$y_{n+1} = y_n + \pi x_n$$

is area-preserving.

[7 marks]

1.7) State two essential requirements of chaos.

[7 marks]

1.8) Define the number δ introduced by Feigenbaum for the logistic map.

[7 marks]

SECTION B – Answer TWO questions

2) Consider the shift map

$$x_{n+1} = 2x_n \pmod 1$$

As usual mod 1 denotes that only the non-integer part of x is considered. Draw the graph of the map.

[2 marks]

By writing x in binary form find all the fixed points.

[4 marks]

Show that the map has periodic points of all periods, and that all of them are unstable.

[10 marks]

Show that the map has an infinite number of aperiodic orbits.

[6 marks]

By considering the rate of separation of two nearby orbits, show that the map has sensitive dependence on initial conditions.

[8 marks]

3) Define the index of an isolated fixed point in a two-dimensional phase space.

[5 marks]

Find the index of a stable node, an unstable node, and a saddle point.

[9 marks]

If a closed curve C surrounds n isolated fixed points $x_1^*, \ldots, x_n^*$, then

$$I_C = I_1 + I_2 + \ldots + I_n$$

where I_k is the index of x_k^*, for $k = 1, \ldots, n$.

Any closed orbit in the phase space must enclose fixed points whose indices sum to +1. Use this to show that closed orbits are impossible for the system of equations

$$\dot{x} = x\,(3 - x - y)$$
$$\dot{y} = y\left(2 - x - \tfrac{y}{2}\right).$$

where x and y are non-negative.

[Hint: assume that the fixed points are given to be $(0,0)$ = unstable node, $(0,4)$ and $(3,0)$ = stable nodes and $(1,2)$ = saddle point.]

[16 marks]

4) Consider a non-uniform oscillator

$$\dot{\theta} = \omega - a \sin\theta$$

where θ is an angle and ω and a are real constants. Use graphical phase space methods to classify the fixed points for $a > \omega$.

[6 marks]

For $a < \omega$ show that the motion is oscillatory with period T given by

$$T = \int_0^{2\pi} \frac{d\theta}{\omega - a\sin\theta}.$$

[7 marks]

By making the substitution $u = \tan\frac{\theta}{2}$ show that

$$T = 2\int_{-\infty}^{\infty} \frac{du}{\omega u^2 - 2au + \omega}.$$

[10 marks]

Furthermore prove that

$$T = \frac{2\pi}{\sqrt{\omega^2 - a^2}}.$$

[5 marks]

[Hint: Make the substitution $x = u - \frac{a}{\omega}$ and use the integral

$$\int dx\, \frac{1}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}.]$$

What is the significance of the singularity in T as $a \to \omega$?

[2 marks]

5) For the following systems of linear differential equations classify the stability characteristic of the steady state at $(x, y) = (0, 0)$:

(a)

$$\frac{dx}{dt} = -y$$
$$\frac{dy}{dt} = 2x.$$

[10 marks]

(b)

$$\frac{dx}{dt} = 3x + y$$
$$\frac{dy}{dt} = 8x + y$$

[10 marks]

(c)

$$\frac{dx}{dt} = -4x - y$$
$$\frac{dy}{dt} = 6x - y.$$

[10 marks]