King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP3212 Statistical Mechanics

Summer 2004

Time allowed: THREE Hours

Candidates should answer no more than SIX parts of SECTION A, and no more than TWO questions from SECTION B. No credit will be given for answering further questions.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper. Where necessary, a College calculator will have been supplied.

Physical Constants

Permittivity of free space	ϵ_0	$=$	8.854×10^{-12}	$\mathrm{F\,m^{-1}}$
Permeability of free space	μ_0	$=$	$4\pi \times 10^{-7}$	$\mathrm{H\,m^{-1}}$
Speed of light in free space	c	$=$	2.998×10^{8}	$\mathrm{m\,s^{-1}}$
Gravitational constant	G	$=$	6.673×10^{-11}	$\mathrm{N\,m^2\,kg^{-2}}$
Elementary charge	e	$=$	1.602×10^{-19}	C
Electron rest mass	m_{e}	$=$	9.109×10^{-31}	kg
Unified atomic mass unit	m_{u}	$=$	1.661×10^{-27}	kg
Proton rest mass	m_{p}	$=$	1.673×10^{-27}	kg
Neutron rest mass	m_{n}	$=$	1.675×10^{-27}	kg
Planck constant	h	$=$	6.626×10^{-34}	J s
Boltzmann constant	k_{B}	$=$	1.381×10^{-23}	$\mathrm{J\,K^{-1}}$
Stefan-Boltzmann constant	σ	$=$	5.670×10^{-8}	$\mathrm{W\,m^2\,K^{-4}}$
Gas constant	R	$=$	8.314	$\mathrm{J\,mol^{-1}\,K^{-1}}$
Avogadro constant	N_{A}	$=$	6.022×10^{23}	$\mathrm{mol^{-1}}$
Molar volume of ideal gas at STP		$=$	2.241×10^{-2}	$\mathrm{m^3}$
One standard atmosphere	P_0	$=$	1.013×10^{5}	$\mathrm{N\,m^{-2}}$

Free energy F in terms of the N-particle partition function Z_N:

$$F = -k_B T \log Z_N$$

The internal energy U in terms of the N-particle partition function Z_N:

$$U = k_B T^2 \frac{\partial}{\partial T} \log Z_N$$

The entropy S in terms of the N-particle partition function Z_N:

$$S = k_B \log Z_N + k_B T \frac{\partial}{\partial T} \log Z_N$$

The number of photons $n(\omega)\, d\omega$ in a frequency interval $(\omega, \omega + d\omega)$ is given by

$$n(\omega)\, d\omega = \frac{V}{\pi^2 c^3} \omega^2 \frac{d\omega}{e^{\frac{\hbar\omega}{k_B T}} - 1}$$

$$\int_0^\infty dx \frac{x^2}{e^x - 1} = 2\varsigma(3) \approx 2.404$$

SECTION A – Answer SIX parts of this section

1.1) Consider an ideal system of 6 non-interacting spin $\frac{1}{2}$ particles in the absence of an external magnetic field. Derive an expression for the probability that only n of the six spins have spin up.

[7 marks]

1.2) The single-particle density of states $D(k)$ in wave-vector k space in 3- dimensions is given by

$$D(k) = \frac{V}{2\pi^2}k^2$$

where V is the volume of the system and k is the magnitude of $\vec{k}$. From this calculate the single-particle density of states in energy for an excitation of energy

$$\varepsilon(k) = \alpha k^{3/2}.$$

[7 marks]

1.3) The energy levels of the three spin states associated with a particle of spin one have energies ε, 0, and $-\varepsilon$ for a certain system. Calculate the free energy of such a system at temperature T with non-interacting particles at each of N sites.

[7 marks]

1.4) Two identical non-interacting spin $\frac{1}{2}$ fermion particles, with spin up, can occupy any of three single particle states. Two of these states have energy 0, and the other has energy ε. Calculate the partition function when the system is at temperature T.

[7 marks]

1.5) Consider the Universe to be a spherical cavity with radius 10^{26}m and temperature 3K. Hence give an estimate of the number of thermally excited photons in the Universe (see rubric).

[7 marks]

1.6) The Fermi wavevector k_F for electrons in a star is given by

$$k_F = \left(3\pi^2 n\right)^{1/3}$$

where n is the density of electrons in a star.

Calculate k_F for a star of mass 3×10^{30} kg and radius 3×10^7m. Assume that the star is mainly composed of dissociated hydrogen atoms.

[7 marks]

1.7) Use the single particle partition function $Z_{sp} = \sum_i e^{-\beta\varepsilon_i}$ to show that for a system of N particles obeying Boltzmann statistics, the internal energy U is given by

$$U = N k_B T^2 \frac{\partial}{\partial T} \log Z_{sp}.$$

[7 marks]

1.8) State the defining properties of microcanonical, canonical and grand-canonical ensembles.

[7 marks]

SECTION B – Answer TWO questions

2) Describe the concepts of macrostate and microstate.

[7 marks]

A system contains 16 identical but distinguishable particles which can occupy non-degenerate equispaced energy levels of spacing ε. The system is initially in the most probable macrostate with 7 particles in the 0 energy level, 4 particles in the ε energy level , 2 particles in the 2ε energy level , 2 particles in the 3ε energy level and 1 particle in the 4ε energy level. This state has a total energy of 18ε.

(a) Show that there are $16!/(7!4!2!2!)$ microstates in this macrostate.

Hence what is the entropy of the system in units of k_B for this macrostate?

[8 marks]

(b) A further amount of energy ε is added to the system so that the total energy is 19ε. As a result one of the particles that was in the ground state is excited to the first excited state. Show that the change in entropy is $k_B \log(7/5)$.

[8 marks]

(c) By use of the formula $\Delta U = T\Delta S$ find an expression for the temperature of the system.

[7 marks]

3) (i) Show that in a system where the electrons are constrained to move in 2-dimensions the density of states $g_{2-D}(k)$ in wavevector space is

$$g_{2-D}(k)\, dk = 2 \times \frac{Akdk}{2\pi}$$

where A is the area of the system.

[12 marks]

(ii) A gas of N fermions is constrained to move only in a two dimensional region of area A (a situation which pertains in some semiconductor structures such as GaAs/AlGaAs heterostructures). Prove that the Fermi energy of the gas is given by $\pi\hbar^2 N/(mA)$.

[9 marks]

(iii) The electrons in a GaAs/AlGaAs heterostructure have a density of 4×10^{11} cm^{-2}. The electrons act as free particles, but their interaction with the lattice means that they appear to have an effective mass of only 15% of their normal mass m_e. Calculate the Fermi energy of the electrons.

[9 marks]

4) Consider an array of N distinguishable spin-$\frac{1}{2}$ paramagnetic atoms. In the presence of a magnetic field B the energies of the two spin states split by $\pm\mu_B B$ where μ_B is the Bohr magneton.

(i) Derive the single particle partition function for the system.

[10 marks]

(ii) Show that the heat capacity C is given by

$$\frac{N\mu^2 B^2}{k_B T^2}\left(\cosh\frac{\mu B}{k_B T}\right)^{-2}$$

and show that it has a peak at a temperature T_{peak} :

$$T_{peak} = \frac{A\mu B}{k_B}$$

where A is a constant.

[10 marks]

(iii) Show that in the limit of low temperature the heat capacity is of the form

$$C \propto \frac{1}{T^2}e^{-\frac{\theta}{T}}\,\mathrm{as\,T} \to 0.$$

[10 marks]

5) A system consists of three indistinguishable atoms and the macroscopic observable is the mean energy. Each atom can exist in three energy states - the ground state chosen, by convention, to have zero energy, and two excited states, one with energy ϵ and one with energy 2ϵ. The ground state is doubly degenerate $[g\,(0) = 2]$ whereas the excited states are non-degenerate. If the system obeys Bose-Einstein statistics there are states with energies $0, \epsilon, 2\epsilon, 3\epsilon, 4\epsilon, 5\epsilon,$ and 6ϵ which have degeneracies $4, 3, 5, 3, 3, 1,$ and 1 respectively. Verify the values of the degeneracies for the states with energies $0$and ϵ.

[10 marks]

Letting $x = exp\left(-\frac{\epsilon}{k_B T}\right)$ write down expressions for the partition function of the system if it obeys (a) Bose-Einstein statistics (b) Fermi-Dirac statistics and (c) Maxwell-Boltzmann statistics.

[12 marks]

Show that the entropy S (see rubric) in the Bose-Einstein case tends to $k_B \log 4$ in the limit as $T \to 0$ and calculate the entropy in the same limit for the Fermi-Dirac case.

What is the physical cause of this difference?

[8 marks]