King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP2380 ELECTROMAGNETISM

Summer 2003

Time allowed: THREE Hours

Candidates must answer SIX parts of SECTION A, and TWO questions from SECTION B.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper. Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

Permittivity of free space	ε_0	$= 8.854 \times 10^{-12}\,\mathrm{F\,m^{-1}}$
$1/(4\pi\varepsilon_0)$		$= 8.988 \times 10^{9}\,\mathrm{N\,m^2C^{-1}}$
Permeability of free space	μ_0	$= 4\pi \times 10^{-7}\,\mathrm{H\,m^{-1}}$
Speed of light in free space	c	$= 2.998 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Planck constant	h	$= 6.626 \times 10^{-34}\,\mathrm{J\,s}$
$h/2\pi =$	$\hbar$	$= 1.054 \times 10^{-34}\,\mathrm{J\,s}$
Electronic charge	e	$= 1.602 \times 10^{-19}\,\mathrm{C}$
Electron rest mass	m_e	$= 9.109 \times 10^{-31}\,\mathrm{kg}$
Electron rest energy	m_ec^2	$= 0.511\,\mathrm{MeV}$

SECTION A – Answer SIX parts of this section

1.1) Consider two conducting spheres, one of radius 6 cm and the other of radius 12 cm. Each has a positive charge of 30 nC and they are very far apart. If the spheres are connected by a conducting wire, find the direction of flow of charge along the wire, and the magnitude of the charge that flows between the spheres.

[7 marks]

1.2) Consider the electromagnetic wave

$$\mathbf{E} = E_0 \cos(\omega(az - t))\,\mathbf{i} + E_0 \sin(\omega(az - t))\,\mathbf{j}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the x and y directions respectively, and E_0 and a are constants.

Calculate the x, y and z components of the time-varying magnetic flux density $\mathbf{B}$ associated with this wave.

[7 marks]

1.3) At a point within a linear isotropic dielectric, the relative permittivity ε_r is 3 and the electric field strength is $10^5\,\mathrm{V\,m^{-1}}$. For the medium at this specific point, find:

(a) the electric susceptibility,

(b) the magnitude of the electric polarisation,

(c) the magnitude of the electric displacement.

[7 marks]

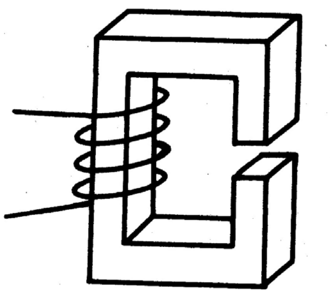
1.4) A 100 pF capacitor is charged to a potential difference of 100 V. After the charging is complete, the battery is disconnected. This capacitor is then connected in parallel across another, uncharged, capacitor, and the final potential difference across the capacitor plates is found to be 30 V. Calculate:

(a) The capacitance of the second capacitor.

(b) The total energy stored by the two capacitors after connection.

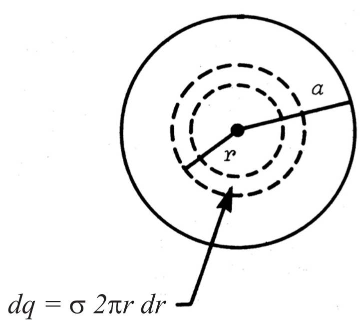
[7 marks]

1.5) The electromagnet shown in the figure has an iron core with magnetic permeability $\mu = 800\mu_0$. The cross-sectional area of the iron is $100\,\text{cm}^2$. The total length of the path around the iron and across the 2 cm gap is 200 cm. By neglecting the bulging of the magnetic field lines out of the gap, estimate how many ampere-turns will be required in the windings to establish a **B**-field of 0.5 T in the gap.



[7 marks]

1.6) An insulating circular disk of radius a has a uniformly distributed static surface charge density σ. The disk rotates about a perpendicular axis through its centre with an angular velocity ω. By considering concentric rings of charge of thickness dr, derive an expression for the magnetic field at the centre of the disk.



[7 marks]

1.7) A spherical drop of rainwater carrying a charge of 30 pC has a potential of 500 V at its surface. Show that the radius of the drop is 5.4×10^{-4} m.

If two such drops, of the same charge and radius, combine to form a single spherical drop, what would be the potential on the surface of the newly formed drop?

[7 marks]

1.8) Assume that the hydrogen atom can be modelled as an electron moving in a circular orbit around a central proton. The orbit radius is

$$a_0 = \frac{\hbar^2}{m_e e^2},$$

and the electron has a speed $v = e^2/\hbar$.

Calculate the current that is equivalent to this circulating charge, and hence find the magnetic field strength at the proton due to the orbital motion of the electron.

[7 marks]

SECTION B – Answer TWO questions

2 (a) A total charge Q is distributed uniformly over the surface of a sphere of radius a. Show that the electrostatic potential energy associated with this charge distribution is

$$U = \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0 a}$$

[6 marks]

(b) Show that when the same charge Q is distributed uniformly throughout the volume of a sphere, also of radius a, the electrostatic potential energy is given by

$$U = \frac{3}{5}\frac{Q^2}{4\pi\varepsilon_0 a}$$

[12 marks]

(c) Estimate the "radius" of the electron by assuming that its rest-mass energy is exclusively electrostatic in origin, and that its charge is uniformly distributed over its surface.

[4 marks]

(d) Estimate the "radius" of the electron by assuming that its rest-mass energy is exclusively electrostatic in origin, and that its charge is uniformly distributed throughout its volume.

[4 marks]

(e) Compare the previous two results with the so-called "classical" electron radius given by the expression

$$\frac{e^2}{4\pi\varepsilon_0 m_e c^2}$$

[4 marks]

3 (a) State the relationship between the electric field $\mathbf{E}$ and the electric potential V. Consider a uniform electric field $\mathbf{E} = (0, 0, E_0)$ in cartesian coordinates. Derive an expression for the corresponding electric potential in spherical polar coordinates.

[5 marks]

(b) A sphere of radius a made of an ideal conducting material carrying a total surface charge Q is placed in the field. Assume the origin of the coordinate system is at the centre of the sphere, and that the zero of the potential V lies in the $x - y$ plane, where x and $y \to \infty$. Explain why the potential at the surface of the sphere is unaffected by the presence of the applied field $\mathbf{E}$.

[5 marks]

(c) Sketch the $\mathbf{E}$-field lines near the sphere, and briefly explain your diagram.

[4 marks]

(d) Show that the electrostatic potntial V distribution outside the sphere will satisfy the Laplace equation. The general solution of the Laplace equation in spherical polar coordinates can be written as

$$V(r, \theta) = \sum_{n=0}^{\infty} \left(A_n r^n + B_n r^{-n-1}\right) P_n(\cos\theta).$$

Hence show that the electrostatic potential distribution outside the sphere is given by

$$V(r, \theta) = \frac{Q}{4\pi\varepsilon_0 r} - \left(1 - \frac{a^3}{r^3}\right) r E_0 \cos\theta$$

[10 marks]

Note: You may assume that

$$P_0(\cos\theta) = 1, \qquad P_1(\cos\theta) = \cos\theta$$

(e) Find the surface charge density $\sigma(\theta)$.

[6 marks]

4 (a) Write down Maxwell's equations for a vacuum in which there is no free charge density and no free current density.

[4 marks]

(b) Assume that the following electric and magnetic fields satisfy Maxwell's equations as given in the answer to part (a):

$$\mathbf{E}(\mathbf{r}) = f(\mathbf{k}.\mathbf{r} - \omega t)\,\mathbf{e} \quad \text{and} \quad \mathbf{B}(\mathbf{r}) = g(\mathbf{k}.\mathbf{r} - \omega t)\,\mathbf{b}$$

$\mathbf{e}$ and $\mathbf{b}$ are constant unit vectors, $\mathbf{k}$ is a constant vector, ω is a scalar constant, and $f(x)$ and $g(x)$ are doubly differentiable functions of one variable. By considering the differential form of Gauss' law for $\mathbf{E}$ and for $\mathbf{B}$, show that the vectors $\mathbf{e}$ and $\mathbf{b}$ must be perpendicular to the vector $\mathbf{k}$.

[6 marks]

(c) By considering expressions for $\partial\mathbf{E}/\partial t$ and $\partial\mathbf{B}/\partial t$, show that $\mathbf{B}$ must be perpendicular to both $\mathbf{E}$ and $\mathbf{k}$.

[4 marks]

(d) Show that $k^2 = \mu_0\varepsilon_0\omega^2$, where $k = |\mathbf{k}|$

[8 marks]

(e) Show that $\mathbf{B} = \mathbf{k} \times \mathbf{E}/\omega$. (Ignore any constant or static fields.)

[4 marks]

(f) Show that if $k^2 = \mu_0\varepsilon_0\omega^2$, then f and g each satisfy the wave equation.

[4 marks]

Note: You may use the following vector identity:

$$\nabla \times \nabla \times \mathbf{F} = \nabla(\nabla.\mathbf{F}) - \nabla^2\mathbf{F}$$

5 (a) Consider a single circular loop of wire, of radius R carrying a current I. Assume that the loop is in the $y-z$ plane defined by $x = 0$. Starting from the Biot-Savart law show that the magnetic field $\mathbf{B}$ at an arbitrary point x along a perpendicular axis through the centre of the loop is given by the expression

$$\mathbf{B}(x) = \frac{\mu_0 I R^2}{2\left(R^2 + x^2\right)^{\frac{3}{2}}}\,\mathbf{i}$$

where $\mathbf{i}$ is a unit vector along the x axis.

[16 marks]

(b) Use this result to show that the magnetic field at the end of a solenoid of radius R, consisting of N tightly wound turns along a length L, is given by

$$\frac{\mu_0 N I}{2\left(R^2 + L^2\right)^{\frac{1}{2}}}$$

[8 marks]

(c) Calculate the field at the centre of the same solenoid. What is the central field value for a very long solenoid ($L \gg R$)?

[6 marks]