King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/2201 INTRODUCTORY QUANTUM MECHANICS

Summer 1999

Time allowed: THREE Hours

Candidates should answer SIX parts of SECTION A, and TWO questions from SECTION B.

Separate answer books must be used for each Section of the paper.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

Pauli matrices are given by

$$\boldsymbol{S}_x = \frac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{S}_y = \frac{1}{2}\hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{S}_z = \frac{1}{2}\hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

SECTION A – Answer SIX parts of this section

1.1) A particle bound in one dimension is described by the normalised wave function

$$\psi(x) = \begin{cases} \sqrt{\frac{2}{L}} e^{ikx} \cos\frac{3\pi x}{L}, & |x| \leq L/2, \\ 0, & \text{elsewhere}. \end{cases}$$

Show that the probability of finding the particle between $x = 0$ and $x = L/4$ is approximately 0.2.

Note: $\cos 2\theta = 2\cos^2\theta - 1$.

[7 marks]

1.2) In three dimensions, the time-dependent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{r})\psi = i\hbar\frac{\partial\psi}{\partial t}.$$

Which variables is ψ a function of ? What do the symbols $\psi, \mathbf{r}, t, m, \hbar, V, i, \nabla^2$ and ∂ in the equation represent? Write the equation in terms of the Hamiltonian operator $\boldsymbol{H}$ for the system, and interpret the various terms in $\boldsymbol{H}$.

[7 marks]

1.3) The possible energies of a particle in a cubic box of side a are given by

$$E_{n_1,n_2,n_3} = ({n_1}^2 + {n_2}^2 + {n_3}^2)\epsilon,$$

where n_1, n_2, n_3 are positive integers and ϵ is a constant. Find the energy of the ground state and the energy of the next lowest *non-degenerate* excited level in terms of the energy ϵ. How many degenerate levels lie between the lowest and next lowest non-degenerate levels, and what are their degeneracies?

[7 marks]

1.4) A particle of mass m and energy E is bound ($E < 0$) in an attractive one-dimensional square-well potential

$$V(x) = \begin{cases} -V_0\,, & |\,x\,| < a\,, \\ 0\,, & |\,x\,| \geq a\,. \end{cases}$$

Show that, in the inner region, the Schrödinger equation has oscillatory-type solutions and, in the outer regions, exponential-type solutions.

[7 marks]

1.5) Use the *correspondence principle* to derive representation-free operators for the Cartesian components of the angular momentum operator $\boldsymbol{L}$. Hence, write down a Cartesian expression for $\boldsymbol{L}_z$ in the Schrödinger representation.

[7 marks]

1.6) The quantum numbers n, ℓ, m_ℓ and m_s appear in the theoretical treatment of the hydrogen atom. Give their allowed values and explain briefly which physical properties are determined by them.

[7 marks]

1.7) Calculate the eigenvalues of the operator $\boldsymbol{S}_y$ representing the $y-$component of the spin angular momentum of a spin-$\frac{1}{2}$ particle. Normalise the corresponding eigenvectors

$$\alpha_y = \begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \beta_y = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

and verify that they are orthogonal.

[7 marks]

1.8) At a given instant, a quantum harmonic oscillator is in a state described by the normalised wave function

$$\psi(x) = \frac{\sqrt{3}}{2} u_0(x) + \frac{1}{2} u_1(x)\,,$$

where $u_n(x)$ is the normalised energy eigenfunction of the oscillator corresponding to an eigenvalue $E_n = (n + \frac{1}{2})\hbar\omega$, $n = 0, 1, 2, \ldots$. Calculate the expectation value of the energy of the oscillator in the state ψ.

[7 marks]

SECTION B – Answer TWO questions

2) In spherical polar coordinates (r, θ, ϕ), the components of the orbital angular momentum operator are given by

$$\boldsymbol{L}_x = i\hbar(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}),$$
$$\boldsymbol{L}_y = i\hbar(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}),$$
$$\boldsymbol{L}_z = -i\hbar\frac{\partial}{\partial\phi}.$$

Using this representation, prove the commutation relation

$$[\boldsymbol{L_z}, \boldsymbol{L_x}] = i\hbar\boldsymbol{L_y}$$

[12 marks]

and, by cyclic interchange of the variables, write down the commutators that equal $i\hbar\boldsymbol{L_z}$ and $i\hbar\boldsymbol{L_x}$.

[3 marks]

What are the physical implications of the commutation relations for angular momentum?

[3 marks]

Show that the *spherical harmonic*

$$Y_{2,2}(\theta, \phi) = \sin^2\theta e^{2i\phi}$$

is an eigenfunction of the operator $\boldsymbol{L}_z$ and determine the corresponding eigenvalue.

[4 marks]

The spherical harmonics $Y_{\ell,m}$ are simultaneous eigenfunctions of $\boldsymbol{L}_z$ and $\boldsymbol{\mathcal{O}}$ with eigenvalues $\mu\hbar$ and $\lambda\hbar^2$, respectively. What is the operator $\boldsymbol{\mathcal{O}}$? Give the values of μ and λ in terms of ℓ and m.

[4 marks]

Give a simple physical argument which constrains the values of m for a given value of ℓ.

[4 marks]

3) The normalised energy eigenfunction for the ground state of the hydrogen atom has the form

$$u(r) = A \exp(-r/a_0)\,,$$

where a_0 is the Bohr radius and A is a normalisation constant. Derive an expression for the probability, $P(r)\mathrm{d}r$, that the electron lies within a spherical shell with radii r and $r + \mathrm{d}r$.

[6 marks]

Prove that the normalisation constant

$$A = (\pi {a_0}^3)^{-\frac{1}{2}}\,.$$

[6 marks]

Deduce the most probable value of the radial coordinate r.

[6 marks]

Finally, calculate the mean value of r

[6 marks]

and its standard deviation, Δr.

[6 marks]

Note: You will find the following integral useful,

$$\int_0^\infty e^{-\alpha r} r^n \,\mathrm{d}r = \frac{n!}{\alpha^{n+1}}\,,$$

where the constant $\alpha > 0$ and the integer $n > -1$.

4) A beam of particles of mass m and energy E is incident from $x < 0$ upon a potential step at $x = 0$ of height V_0 $(< E)$. Let

$$k^2 = \frac{2mE}{\hbar^2}\,, \qquad \kappa^2 = \frac{2m}{\hbar^2}(E - V_0)\,, \qquad \mu = \frac{\kappa}{k}\,,$$

and the incident particles be represented by the wavefunction e^{ikx}. Calculate the reflection coefficient $\mathcal{R}$ and the transmission coefficient $\mathcal{T}$ as functions of μ.

[24 marks]

Hence demonstrate explicitly that $\mathcal{R} + \mathcal{T} = 1$. What is the physical interpretation of this equation?

[6 marks]

5) Consider a beam of neutral spin-$\frac{1}{2}$ particles, travelling along the $y-$axis, that has passed through the positive channel of a Stern-Gerlach SGZ-apparatus. It then passes through an SGθ-apparatus orientated to measure the spin component in the xz-plane at an angle θ to the positive z-axis. The operator $\boldsymbol{S}_\theta$ representing the component of spin angular momentum in this direction is given by

$$\boldsymbol{S}_\theta = \frac{1}{2}\hbar \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} .$$

Verify that the eigenvectors of $\boldsymbol{S}_\theta$ are

$$\alpha_\theta = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} , \qquad \beta_\theta = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

and find the corresponding eigenvalues.

[7 marks]

Calculate the relative intensities of the exit beams

[14 marks]

and interpret the results when $\theta = 0, \frac{\pi}{2}$ and π.

[9 marks]

Note: You may find the following relations useful,

$$\begin{aligned} \sin(A-B) &= \sin A \cos B - \cos A \sin B \,, \\ \cos(A-B) &= \cos A \cos B + \sin A \sin B \,, \\ \sin 2A &= 2 \sin A \cos A \,, \\ \cos 2A &= 1 - 2\sin^2 A = 2\cos^2 A - 1 \,. \end{aligned}$$