King's College London

UNIVERSITY OF LONDON

This paper is part of an examination of the College counting towards the award of a degree. Examinations are governed by the College Regulations under the authority of the Academic Board.

B.Sc. EXAMINATION

CP/2201 INTRODUCTORY QUANTUM MECHANICS

Summer 1996

Time allowed: THREE Hours

Candidates should answer SIX parts of SECTION A, and TWO questions from SECTION B.

Separate answer books must be used for each Section of the paper.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper. Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

Values of physical constants

mass of neutron	$m_{\rm n}$	$= 1.675 \times 10^{-27}$ kg
elementary charge	e	$= 1.602 \times 10^{-19}$ C
Planck constant	h	$= 6.626 \times 10^{-34}$ Js
speed of light	c	$= 2.998 \times 10^{8}$ ms^{-1}

Pauli matrices are given by

$$\boldsymbol{S}_x = \frac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{S}_y = \frac{1}{2}\hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{S}_z = \frac{1}{2}\hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

SECTION A – Answer SIX parts of this section

1.1) Explain what is meant by the italicised words in the following statement: the wave functions corresponding to *non-degenerate* bound states are *orthogonal* and *normalized.*

How is the probability of finding a particle in a particular region of space related to the wave function?

[7 marks]

1.2) Explain briefly what is meant by the *correspondence principle* and by the *complementarity principle.*

[7 marks]

1.3) Show that

$$u(x) = e^{-\frac{1}{2}x^2}$$

is an eigenfunction of the operator

$$\boldsymbol{A} = \frac{\partial^2}{\partial x^2} - x^2$$

and find the corresponding eigenvalue.

[7 marks]

1.4) The observables A and B are *compatible.* What does this imply about the eigenfunctions of the corresponding operators $\boldsymbol{A}$ and $\boldsymbol{B}$? Prove that $\boldsymbol{A}$ and $\boldsymbol{B}$ commute.

What are the physical implications of compatibility? Give one example of a compatible pair of observables and one example of an *incompatible* pair.

[7 marks]

1.5) A quantum harmonic oscillator is in a state described by the normalized wave function

$$\psi(x) = \sqrt{\frac{1}{3}}u_0(x) + \sqrt{\frac{1}{6}}u_2(x) + \sqrt{\frac{1}{2}}u_4(x) ,$$

where $u_n(x)$ is the nth normalized energy eigenfunction of the oscillator corresponding to an eigenvalue $E_n = (n+\frac{1}{2})\hbar\omega, n = 0, 1, 2, \ldots$. What are the possible results of a measurement of the energy of this system and what are their relative probabilities? Using these probabilities, show that the expectation value of the energy of the oscillator is $\frac{17}{6}\hbar\omega$.

[7 marks]

1.6) State and explain Heisenberg's uncertainty principle for a particle moving in one dimension.
A discharge tube produces an emission line of wavelength 500 nm with an intrinsic width of 2×10^{-4} nm. Assuming that the transition is from an excited state to the ground state, estimate the life-time of the excited state.

[7 marks]

1.7) An electron is in the spin state

$$\psi = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} .$$

Find the expectation value in this state of the spin component $\boldsymbol{S}_y$. Write down the two normalized eigenvectors of the operator $\boldsymbol{S}_z$ and express ψ as a linear combination of them.

[7 marks]

1.8) Explain the significance of the quantum numbers n, ℓ, m_ℓ and m_s used to describe the bound states of the hydrogen atom.

[7 marks]

SECTION B – Answer TWO questions

2) Derive the expression

$$\boldsymbol{S}_\theta = \frac{1}{2}\hbar \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

for the operator representing the component of spin angular momentum in the direction aligned at an angle θ to the positive z-axis in the xz-plane.

[8 marks]

Show that $\boldsymbol{S}_\theta$ has eigenvalues $\pm\frac{1}{2}\hbar$,

[5 marks]

with the normalized eigenvector

$$\beta_\theta = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

corresponding to $-\frac{1}{2}\hbar$.

[5 marks]

A beam of neutral spin-$\frac{1}{2}$ particles travelling in the y-direction passes through a Stern-Gerlach apparatus. The spins in the two exit beams are in the eigenstates of $\boldsymbol{S_z}$ and all particles in the entry beam are in the eigenstate β_θ. Calculate the relative intensities of the exit beams.

[12 marks]

3) Describe **two** of the following:

(i) the Stern-Gerlach experiment and the way in which it provides evidence for the existence of electron spin,

[15 marks]

(ii) an effect which demonstrates the particle aspects of electromagnetic radiation,

[15 marks]

(iii) an experiment which provides evidence for the existence of matter waves,

[15 marks]

(iv) a physical process in which barrier penetration is important.

[15 marks]

4) A neutron of mass m_{n} and energy E is bound ($E < 0$) in an attractive one-dimensional square-well potential

$$V(x) = \begin{cases} -V_0\,, & \mid x \mid < a\,, \\ 0\,, & \mid x \mid > a\,. \end{cases}$$

Show that, in the inner region, the Schrödinger equation has oscillatory-type solutions and, in the outer regions, exponential-type solutions.

[8 marks]

Describe *qualitatively* how the unknown constants of integration are determined from the boundary conditions.

[5 marks]

Explain why non-zero solutions in the outer regions are classically forbidden.

[2 marks]

Assume that the bound-state solutions fall into two sets depending on whether they have even or odd parity. The allowed energy eigenvalues are known to be given by the equations

$$\sqrt{\frac{p^2}{\xi^2} - 1} = \begin{cases} -\cot\xi\,, & \text{odd parity} \\ \tan\xi\,, & \text{even parity} \end{cases}$$

where

$$\xi = \left[\frac{2m_{\mathrm{n}}}{\hbar^2} a^2 (V_0 - \mid E \mid)\right]^{\frac{1}{2}}$$

and the parameter p is defined by

$$p^2 = \frac{2m_{\mathrm{n}}}{\hbar^2}\, V_0\, a^2\,.$$

Show that there is always at least one bound state of even parity.

[5 marks]

If $a = 10^{-5}$ nm, find the minimum well-depth in MeV that would allow at least **two** bound states of **even** parity.

[10 marks]

5) In spherical polar coordinates (r, θ, ϕ), the components of the orbital angular momentum operator are given by

$$\boldsymbol{L}_x = i\hbar(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}),$$
$$\boldsymbol{L}_y = i\hbar(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}),$$
$$\boldsymbol{L}_z = -i\hbar\frac{\partial}{\partial\phi}.$$

Using this representation, prove the commutation relation

$$[\boldsymbol{L_z}, \boldsymbol{L_x}] = i\hbar\boldsymbol{L_y}.$$

[9 marks]

Compare this result with the commutation relation for the linear momentum operators $\boldsymbol{p}_z$ and $\boldsymbol{p}_x$.

[4 marks]

Show that the function

$$Y_{2,1}(\theta, \phi) = \cos\theta\sin\theta e^{i\phi}$$

is an eigenfunction of the operator $\boldsymbol{L}_z$ and determine the corresponding eigenvalue.

[4 marks]

Evaluate the function

$$(\boldsymbol{L}_x - i\boldsymbol{L}_y)Y_{2,1}(\theta, \phi)$$

and show that it is proportional to

$$Y_{2,0}(\theta, \phi) = 3\cos^2\theta - 1.$$

[9 marks]

Show that $(\boldsymbol{L}_x - i\boldsymbol{L}_y)Y_{2,1}(\theta, \phi)$ is an eigenfunction of $\boldsymbol{L}_z$ with zero eigenvalue.

[4 marks]