King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/2201 INTRODUCTORY QUANTUM MECHANICS

Summer 2001

Time allowed: THREE Hours

Candidates should answer SIX parts of SECTION A, and TWO questions from SECTION B.

Separate answer books must be used for each Section of the paper.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper. Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

Pauli matrices are given by

$$\boldsymbol{S}_x = \frac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{S}_y = \frac{1}{2}\hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{S}_z = \frac{1}{2}\hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

SECTION A – Answer SIX parts of this section

1.1) The unnormalized energy eigenfunction of the first excited state of a particle moving in a one-dimensional harmonic oscillator potential is

$$u(x) = xe^{-\frac{1}{2}\alpha^2 x^2},$$

where α is a constant. Normalize this eigenfunction and write down an integral which gives the probability that the particle is in the interval $|x| \leq 1/\alpha$.

Note:

$$\int_{-\infty}^{\infty} x^{2n} e^{-\alpha^2 x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}(2n)!}{2^{2n} n!\,\alpha^{2n+1}}, \qquad n = 1, 2, 3, \ldots.$$

[7 marks]

1.2) An operator $\boldsymbol{A}$ satisfying

$$\int \psi_1^* \boldsymbol{A} \psi_2 \,\mathrm{d}^3\mathbf{r} = \int \psi_2 \boldsymbol{A}^* \psi_1^* \,\mathrm{d}^3\mathbf{r},$$

where $\psi_1(\boldsymbol{r})$ and $\psi_2(\boldsymbol{r})$ are any well-behaved functions of $\boldsymbol{r}$ which vanish at ∞, is said to be *hermitian.* Prove that the operator $-i\hbar\partial/\partial x$ is hermitian. What observable does the operator represent?

[7 marks]

1.3) At time $t = 0$, a quantum harmonic oscillator is in a state described by the normalized wave function

$$\psi(x, 0) = \sqrt{\frac{1}{5}}u_0(x) + \sqrt{\frac{1}{2}}u_2(x) + \sqrt{\frac{3}{10}}u_3(x),$$

where $u_n(x)$ is the nth normalized energy eigenfunction of the oscillator corresponding to an eigenvalue $E_n = (n+\frac{1}{2})\hbar\omega$, $n = 0, 1, 2, \ldots$. What are the possible results of a measurement of the energy of this system and what are their relative probabilities? Using these probabilities, show that the expectation value of the energy of the oscillator at $t = 0$ is $\frac{12}{5}\hbar\omega$.

[7 marks]

1.4) Define the parity operator $\boldsymbol{P}$ and prove that the eigenvalues of $\boldsymbol{P}$ are ± 1.

[7 marks]

1.5) The possible energies of a particle in a box with sides $(3a, 3a, a)$ are given by

$$E_{n_1,n_2,n_3} = ({n_1}^2 + {n_2}^2 + 9{n_3}^2)\epsilon \,,$$

where n_1, n_2, n_3 are positive integers and ϵ is a constant. What is the energy of the ground state in terms of the energy ϵ. Show that the energies of the two lowest *non-degenerate* excited levels are 17ϵ and 27ϵ. How many degenerate levels lie between the ground state and the second non-degenerate excited state, and what are their degeneracies?

[7 marks]

1.6) Write down Schrödinger's equation of motion for a particle in a static potential and confined to the $x-$axis. Show how the spatial (x) and temporal (t) variables can be separated and derive a particular solution for the time dependence.

[7 marks]

1.7) An electron is in the unnormalized spin state

$$\psi = \begin{pmatrix} 2 \\ i \end{pmatrix} .$$

Normalize ψ and find the expectation value of the spin component $\boldsymbol{S}_z$ in this state. What is the probability that a measurement of $\boldsymbol{S}_z$ will correspond to the electron being in the spin-up state?

[7 marks]

1.8) The spherical harmonics $Y_{\ell,m}$ are simultaneous eigenfunctions of $\boldsymbol{L}_z$ and $\boldsymbol{O}$ with eigenvalues $\mu\hbar$ and $\lambda\hbar^2$, respectively. What is the operator $\boldsymbol{O}$? Give the values of μ and λ in terms of ℓ and m. What are the allowed values of ℓ and m? Give a simple physical argument that constrains the values of m for a given value of ℓ.

[7 marks]

SECTION B – Answer TWO questions

2) A beam of neutral spin-$\frac{1}{2}$ particles travelling in the $y-$direction passes through a Stern-Gerlach apparatus. The spins in the two exit beams are in the eigenstates of $\boldsymbol{S}_x$. All particles in the entry beam are in the eigenstate α_θ corresponding to the eigenvalue $+\frac{1}{2}\hbar$ for the component of spin angular momentum in a direction aligned at an angle θ to the positive $z-$axis in the $xz-$plane. Calculate the appropriate spin operator $\boldsymbol{S}_\theta$

[8 marks]

and hence show that

$$\alpha_\theta = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} .$$

[6 marks]

Prove that the relative intensities of the exit beams are $\frac{1}{2}(1 \pm \sin\theta)$.

[16 marks]

3) A beam of particles of mass m and energy E is incident from $x < 0$ upon a potential step at $x = 0$ of *depth* $-V_0$. Let

$$k^2 = \frac{2mE}{\hbar^2}\,, \qquad \kappa^2 = \frac{2m}{\hbar^2}(E + V_0)\,, \qquad \mu = \frac{\kappa}{k}\,,$$

and the incident particles be represented by the wavefunction e^{ikx}. Calculate the reflection coefficient $\mathcal{R}$ and the transmission coefficient $\mathcal{T}$ as functions of μ,

[16 marks]

and compare your answers with the classical results.

[4 marks]

Compare the particles' momentum on either side of $x = 0$. Is the comparison consistent with the classical result?

[6 marks]

What happens to $\mathcal{R}$ and $\mathcal{T}$ in the limit of a low-energy ($E \ll V_0$) incident beam?

[4 marks]

4) Consider a particle confined by the infinite square-well potential

$$V(x) = \begin{cases} 0\,, & 0 < x < L\,, \\ \infty\,, & \text{elsewhere.} \end{cases}$$

The time-independent energy eigenfunctions are of the form

$$u(x) = A \sin kx\,.$$

(i) By considering the boundary conditions, find the allowed values of k;

[5 marks]

(ii) By applying the normalization condition, show that $A = \sqrt{2/L}$;

[5 marks]

(iii) Show explicitly that different eigenfunctions are orthogonal;

[5 marks]

(iv) By substituting the eigenfunctions into Schrödinger's equation, determine the energy eigenvalues;

[5 marks]

(v) Calculate the mean value of x for each eigenfunction;

[5 marks]

(vi) Suppose that, at a certain time, the particle is in the state

$$\psi(x) = \sqrt{\frac{1}{L}}\sin\frac{\pi x}{L} + \frac{1}{2}\sqrt{\frac{1}{L}}\sin\frac{2\pi x}{L} + \frac{1}{2}\sqrt{\frac{3}{L}}\sin\frac{4\pi x}{L}\,.$$

Determine the probability that, on measuring the particle's energy, one obtains a value corresponding to (a) the ground state, and (b) the first excited state.

[5 marks]

Note. You will find the following integrals useful:

$$\int_0^{\pi} \sin mx\, \sin nx\, \mathrm{d}x = \frac{\pi}{2}\delta_{m,n}$$

$$\int_0^{\pi} x \sin^2 mx\, \mathrm{d}x = \frac{\pi^2}{4}$$

where $m, n = 1, 2, 3, \ldots$ and $\delta_{m,n}$ is the Kronecker delta.

5) In spherical polar coordinates (r, θ, ϕ), the components of the orbital angular momentum operator are given by

$$\boldsymbol{L}_x = i\hbar(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}),$$
$$\boldsymbol{L}_y = i\hbar(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}),$$
$$\boldsymbol{L}_z = -i\hbar\frac{\partial}{\partial\phi}.$$

Using this representation, prove the commutation relation

$$[\boldsymbol{L_y}, \boldsymbol{L_z}] = i\hbar\boldsymbol{L_x}.$$

[10 marks]

Prove the corresponding relation between the components of the spin angular momentum.

[8 marks]

State the commutation relation for the corresponding linear momentum operators.

[2 marks]

Which of the above pairs of operators are described as *compatible* and which as *incompatible*? What are the physical implications of these terms?

[5 marks]

Indicate why a representation in terms of differential operators does not exist for spin angular momentum.

[5 marks]