King's College London

UNIVERSITY OF LONDON

This paper is part of an examination of the College counting towards the award of a degree. Examinations are governed by the College Regulations under the authority of the Academic Board.

B.Sc. EXAMINATION

CP/2201 INTRODUCTORY QUANTUM MECHANICS

Summer 2000

Time allowed: THREE Hours

Candidates should answer SIX parts of SECTION A, and TWO questions from SECTION B.

Separate answer books must be used for each Section of the paper.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper. Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

2000 ©King's College London

Values of physical constants

mass of neutron	m_{n}	$= 1.675 \times 10^{-27}$ kg
Planck constant	h	$= 6.626 \times 10^{-34}$ J s
electron volt	1 eV	$= 1.602 \times 10^{-19}$ J

Pauli matrices are given by

$$\boldsymbol{S}_x = \frac{1}{2}\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{S}_y = \frac{1}{2}\hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{S}_z = \frac{1}{2}\hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

SECTION A – Answer SIX parts of this section

1.1) A particle bound in one dimension is confined to the interval $|x| \leq L$ and is described by the wave function

$$u(x) = Ae^{-2ikx} \sin \frac{\pi x}{2L}.$$

Determine the normalization constant A.

Note: $$\cos 2\theta = 1 - 2\sin^2\theta\,.$$

[7 marks]

1.2) Show that

$$u(x) = e^{-\frac{1}{2}x^2}$$

is an eigenfunction of the operator

$$\boldsymbol{A} = \frac{d^2}{dx^2} - x^2$$

and find the corresponding eigenvalue.

[7 marks]

1.3) Consider the equation

$$-\frac{\hbar^2}{2m}\nabla^2 u + Vu = Eu\,.$$

What is this equation called and what do the symbols represent? Explain (without proof) how solutions of this equation may be used to construct stationary state solutions of the Schrödinger wave equation.

[7 marks]

1.4) A quantum harmonic oscillator is in a state described by the normalized wave function

$$u(x) = \sqrt{\frac{2}{3}}u_0(x) + \frac{\sqrt{2}}{3}u_3(x) + \frac{1}{3}u_5(x) ,$$

where $u_n(x)$ is the nth normalized energy eigenfunction of the oscillator corresponding to an eigenvalue $E_n = (n + \frac{1}{2})\hbar\omega_c$, $n = 0, 1, 2, \ldots$. What are the possible results of a measurement of the energy of this system and what are their relative probabilities? Using these probabilities, show that the expectation value of the energy of the oscillator is $\frac{31}{18}\hbar\omega_c$.

[7 marks]

1.5) The possible energies of a particle in a box with sides $(L, L, 2L)$ are given by

$$E_{n_1,n_2,n_3} = (4{n_1}^2 + 4{n_2}^2 + {n_3}^2)\epsilon ,$$

where n_1, n_2, n_3 are positive integers and ϵ is a constant. What is the energy of the ground state in terms of the energy ϵ? Find the energy and degeneracy of the lowest *degenerate* level. How many non-degenerate levels lie between the ground state and the lowest degenerate level, and what are their energies?

[7 marks]

1.6) Define a hermitian operator and prove that all its eigenvalues are real.

[7 marks]

1.7) An electron is in the un-normalized spin state

$$\psi = \begin{pmatrix} 1 \\ 2 \end{pmatrix} .$$

Normalize ψ and find the expectation value in this state of the spin component $\boldsymbol{S}_y$. What is the probability that a measurement of $\boldsymbol{S}_y$ gives the value $-\frac{1}{2}\hbar$?

[7 marks]

1.8) The *generalised uncertainty principle* for two observables represented by the operators **A** and **B** is

$$\Delta_A \, \Delta_B \geq \frac{1}{2} \; |< [\boldsymbol{A}, \boldsymbol{B}] >| ,$$

where Δ_X is the uncertainty associated with the observable **X**. Use this result to derive the Heisenberg uncertainty relation between position and momentum.

[7 marks]

SECTION B – Answer TWO questions

2) Derive the expression

$$\boldsymbol{S}_\theta = \frac{1}{2}\hbar \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

for the operator representing the component of spin angular momentum in the direction aligned at an angle θ to the positive z-axis in the xz-plane.

[8 marks]

Show that $\boldsymbol{S}_\theta$ has eigenvalues $\pm\frac{1}{2}\hbar$,

[6 marks]

with the normalized eigenvector

$$\alpha_\theta = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

corresponding to $+\frac{1}{2}\hbar$.

[6 marks]

A beam of neutral spin-$\frac{1}{2}$ particles travelling in the y-direction passes through a Stern-Gerlach apparatus. The spins in the two exit beams are in the eigenstates of $\boldsymbol{S_z}$ and all particles in the entry beam are in the eigenstate α_θ. Calculate the relative intensities of the exit beams.

[10 marks]

3) A particle of mass m moves in one dimension in a potential $V(x) = \frac{1}{2}m{\omega_c}^2x^2$. Show that ω_c is the classical angular frequency of vibration.

[4 marks]

In the ground state, the energy eigenvalue is $\frac{1}{2}\hbar\omega_c$ and the normalised energy eigenfunction is

$$u_0(x) = (\alpha/\pi)^{1/4}e^{-\frac{1}{2}\alpha x^2}$$

where $\alpha = m\omega_c/\hbar$. Calculate the expectation values of

(i) the position x, (ii) x^2, (iii) the linear momentum p_x, and (iv) ${p_x}^2$,
for a particle known to be in the lowest energy state immediately prior to these measurements being made.

[16 marks]

Hence, show explicitly that the expectation value of the total energy is equal to the ground state energy eigenvalue.

[5 marks]

Show that the product of the root-mean-square deviations of x and p_x has the minimum value allowed by Heisenberg's uncertainty principle.

[5 marks]

Note. You will find the following integral useful,

$$\int_{-\infty}^{\infty} x^{2n}e^{-\alpha x^2}\,\mathrm{d}x = \begin{cases} (\pi/\alpha)^{1/2}, & n = 0 \\ \frac{1\times3\times5...(2n-1)\pi^{1/2}}{2^n\alpha^{(2n+1)/2}}, & n = 1, 2, 3, \ldots \end{cases}$$

where the constant $\alpha > 0$.

4) A neutron of mass m_{n} and energy E is bound ($E < 0$) in a one-dimensional square-well potential

$$V(x) = \begin{cases} -V_0\,, & |\,x\,| < a\,, \\ 0\,, & |\,x\,| > a\,. \end{cases}$$

Show that, in the inner region, the Schrödinger equation has oscillatory-type solutions and, in the outer regions, exponential-type solutions.

[8 marks]

Describe *qualitatively* how the unknown constants of integration are determined from the boundary conditions.

[5 marks]

Explain classically why the particle is forbidden in the outer regions.

[2 marks]

Assume that the bound-state solutions fall into two sets depending on whether they have even or odd parity. The allowed energy eigenvalues are known to be given by the equations

$$\sqrt{\frac{p^2}{\xi^2} - 1} = \begin{cases} -\cot\xi\,, & \text{odd parity} \\ \tan\xi\,, & \text{even parity} \end{cases}$$

where

$$\xi = \left[\frac{2m_{\mathrm{n}}}{\hbar^2} a^2 (V_0 - |\,E\,|)\right]^{\frac{1}{2}}$$

and the parameter p is defined by

$$p^2 = \frac{2m_{\mathrm{n}}}{\hbar^2} V_0\, a^2\,.$$

Show that there is always at least one bound state of even parity.

[5 marks]

If $a = 5 \times 10^{-6}$ nm, find the minimum well-depth in MeV that would allow the neutron to have at least **one** bound state of **odd** parity.

[10 marks]

5) It can be proved that the commutator of $\boldsymbol{L}_x$ and $\boldsymbol{L}_y$ is equal to $i\hbar\boldsymbol{L}_z$, and that the coordinate variables may be interchanged cyclically. Hence, write down the set of commutation relations between pairs of angular momentum components.

[3 marks]

Compare your results with the corresponding relations between components of the linear momentum.

[3 marks]

What are the physical implications of the above relations?

[4 marks]

Show explicitly that

$$Y_{2,1}(\theta, \phi) = N_{2,1} \cos\theta \sin\theta e^{+i\phi}$$

is an eigenfunction of the operator $\boldsymbol{L}_z$ and hence determine the corresponding eigenvalue.

[6 marks]

The spherical harmonics $Y_{\ell,m}(\theta, \phi)$ are simultaneous eigenfunctions of $\boldsymbol{L}_z$ and $\boldsymbol{L}^2$. Write down the corresponding pair of eigenvalue equations, expressing the eigenvalues in terms of ℓ and m.

[4 marks]

What are the allowed values of ℓ and m?

[2 marks]

Give a simple geometric argument that constrains the values of m for a given value of ℓ.

[4 marks]For a particle in the state $Y_{2,1}(\theta, \phi)$, calculate the angle that the $\boldsymbol{L}$ vector makes with the z-axis.

[4 marks]