King's College London

UNIVERSITY OF LONDON

This paper is part of an examination of the College counting towards the award of a degree. Examinations are governed by the College Regulations under the authority of the Academic Board.

B.Sc. EXAMINATION

CP/2201 Introduction to Quantum Mechanics

Summer 2005

Time allowed: THREE Hours

Candidates should answer all SIX parts of SECTION A, and no more than TWO questions from SECTION B. No credit will be given for answering further questions.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper. Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED
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Physical Constants

Permittivity of free space	ϵ_0	=	8.854×10^{-12}	$\mathrm{F\,m^{-1}}$
Permeability of free space	μ_0	=	$4\pi\times10^{-7}$	$\mathrm{H\,m^{-1}}$
Speed of light in free space	c	=	2.998×10^{8}	$\mathrm{m\,s^{-1}}$
Gravitational constant	G	=	6.673×10^{-11}	$\mathrm{N\,m^2\,kg^{-2}}$
Elementary charge	e	=	1.602×10^{-19}	C
Electron rest mass	m_{e}	=	9.109×10^{-31}	kg
Unified atomic mass unit	m_{u}	=	1.661×10^{-27}	kg
Proton rest mass	m_{p}	=	1.673×10^{-27}	kg
Neutron rest mass	m_{n}	=	1.675×10^{-27}	kg
Planck constant	h	=	6.626×10^{-34}	$\mathrm{J\,s}$
Boltzmann constant	k_{B}	=	1.381×10^{-23}	$\mathrm{J\,K^{-1}}$
Stefan-Boltzmann constant	σ	=	5.670×10^{-8}	$\mathrm{W\,m^{-2}\,K^{-4}}$
Gas constant	R	=	8.314	$\mathrm{J\,mol^{-1}\,K^{-1}}$
Avogadro constant	N_{A}	=	6.022×10^{23}	$\mathrm{mol^{-1}}$
Molar volume of ideal gas at STP		=	2.241×10^{-2}	$\mathrm{m^3}$
One standard atmosphere	P_0	=	1.013×10^{5}	$\mathrm{N\,m^{-2}}$
Bohr magneton	μ_B	=	9.274×10^{-24}	$\mathrm{J\,T^{-1}}$

$$\int_{-\infty}^{\infty}\exp(-\alpha x^2)\,\mathrm{d}x=\sqrt{\frac{\pi}{\alpha}},\qquad \int_{-\infty}^{\infty}x^2\exp(-\alpha x^2)\,\mathrm{d}x=\frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\sin^2\theta=\frac{1-\cos 2\theta}{2}$$

$\int_0^{\pi}\sin(mx)\sin(nx)\mathrm{d}x=\frac{\pi}{2}\delta_{m,n}$. m,n are integers and $\delta_{m,n}$ is the Kronecker δ.

SECTION A – Answer all SIX parts of this section

1.1) The normalised wavefunction of a harmonic oscillator in its ground state is

$$\psi(x) = C \exp\left(-\frac{x^2}{2a^2}\right)$$

where a is a real constant. Show that

$$C = a^{-1/2}\pi^{-1/4}.$$

[7 marks]

1.2) A particle that is confined in a cuboid with sides of length L, L and $2L$ has energies

$$E_{l,m,n} = \left(4l^2 + 4m^2 + n^2\right) A$$

where A is a constant and $l,\ m,\ n$ are positive (non-zero) integers. Giving your answers in terms of the energy A, what is the lowest energy state of the particle, and what is the energy of the lowest *degenerate* state?

[7 marks]

1.3) The normalised state of a rotating molecule is described by a wavefunction

$$\psi(x) = \sqrt{\frac{1}{6}}u_0(x) + \sqrt{\frac{1}{3}}u_1(x) + \sqrt{\frac{1}{2}}u_2(x),$$

where $u_j(x)$ is the j-th normalised energy eigenfunction of the rotator, corresponding to an eigenvalue $E_j = j(j+1)\hbar^2/2I$, $j = 0,\ 1,\ 2, \ldots$, and I is the moment of inertia of the molecule. What are the possible results of a measurement of the energy of this system, and what are their relative probabilities? Using these probabilities, show that the expectation value of the energy of the oscillator is $11\hbar^2/6I$.

[7 marks]

1.4) A particle of mass m is trapped in a one-dimensional region of length L in which the potential energy of the particle is zero. Write down an expression for the uncertainty in the momentum of the particle and use that expression to *estimate* the minimum energy (in Joules) that an electron can have if it is trapped in a region of length $L = 1$ nm.

[7 marks]

1.5) A particle moves along the x axis in a region of zero potential. Show by substitution in the time-independent Schrödinger equation that $\psi(x) = \exp(ikx)$ is a wavefunction for the particle with energy $E = \hbar^2 k^2/2m$. Given that the momentum operator is $-i\hbar d/dx$, what is the momentum of the particle in terms of k?

[7 marks]

1.6) A particle is trapped in a one-dimensional region of zero potential between $x = 0$ and $x = L$. The normalised wavefunctions can be written as

$$\psi(x) = \sqrt{\frac{2}{L}} \sin\left(n\pi x/L\right),$$

where n is an integer. Show that any two states with different values of n are orthogonal.

[7 marks]

SECTION B – Answer TWO questions

2) A particle of mass m moves in a one dimensional potential $V(x) = \frac{1}{2}m\omega^2 x^2$. It performs harmonic oscillations about $x = 0$ with an angular frequency ω. The lowest energy state has the normalised eigenfunction

$$\psi_0(x) = (\alpha/\pi)^{1/4} \exp\left(-\alpha x^2/2\right),$$

where $\alpha = m\omega/\hbar$.

a) Using the wavefunction $\psi_0(x)$, show that the mean-square displacement of the particle is

$$\overline{x^2} = \frac{\hbar}{2m\omega}.$$

[8 marks]

b) Given that the linear-momentum operator is $\hat{p} = -i\hbar d/dx$, and using the wavefunction $\psi_0(x)$, show that the mean-square momentum of the particle is

$$\overline{p^2} = \frac{\hbar m\omega}{2}.$$

[10 marks]

c) Using the results from (a) and (b), or by any other method, show that the energy of this state is $E_0 = \hbar\omega/2$.

[7 marks]

d) What is the product of the root-mean-square displacement and the root-mean-square momentum, and how is this product related to the Heisenburg uncertainty principle.

[5 marks]

3) A particle of mass m is confined in the x dimension by a potential

$$V(x) = 0, \quad 0 < x < a, \text{ and } V(x) = \infty, \quad \text{otherwise.}$$

a) Show that the time-independent wavefunctions in the region $0 < x < a$ can be written in the form

$$\psi(x) = A \sin(kx).$$

[4 marks]

b) Using a suitable boundary condition, show that $k = n\pi/a$, where n is an integer. State, giving a reason, if the value $n = 0$ is allowed.

[6 marks]

c) Normalise the wavefunctions to show that $A = \sqrt{2/a}$ for all values of n.

[5 marks]

d) A 'quantum layer' is made from a thin film of one material, of thickness $a = 2.0$ nm embedded inside two thicker pieces of a different material so that an electron is trapped in the quantum layer. Using m_e for the mass of the electron, what is the energy E_0, in Joules, of the lowest energy state of an electron in the layer?

[8 marks]

e) When the electron is in the lowest energy state, what is the probability that it can be found in the small length $x = 0.995$ to $x = 1.005$ nm ? (Full marks will be given for an exact answer, or for an approximate answer that makes use of the fact that only a small length is being considered.)

[7 marks]

4) A beam of particles, each of mass m and energy E, travels in a region of zero potential energy from $x = -\infty$ in the positive x direction. At $x = 0$ the particles hit a region of lower potential, where they have a potential energy of $V_0 < 0$. This lower potential V_0 continues for all positive x. The wavefunction of the incident particles is represented by $A_0 \exp(-ikx)$, and the wavefunction for the reflected particles by $A_R \exp(+ikx)$, where

$$k^2 = \frac{2mE}{\hbar^2}.$$

a) Write down the Schrödinger equations for the regions $x < 0$ and $x > 0$, and show that the wavefunction for a particle at $x > 0$ can be written as $B \exp(-iKx)$ where B is a constant and $K = \sqrt{2m(E - V_0)/\hbar^2}$.

[6 marks]

b) What is the ratio A_R/A_0 of the amplitudes of the reflected and incident waves? What is the significance of the sign of A_R/A_0 ?

[8 marks]

c) Show that the reflection coefficient R at the potential step is

$$R = \left| \frac{k - K}{k + K} \right|^2.$$

[4 marks]

d) What is the behaviour of R in the limits $E >> |V_0|$ and $E << |V_0|$?

[4 marks]

e) A beam of electrons is accelerated through a potential of 1000 V. It is then accelerated a second time by a step potential of $V_0 = -3000$ V. What is the probability that an electron is reflected by the step-potential? What is the probability that it will be transmitted?

[8 marks]