King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/3630 GENERAL RELATIVITY AND COSMOLOGY

Summer 1999

Time allowed: THREE Hours

Candidates should answer SIX parts of SECTION A, and TWO questions from SECTION B.

Separate answer books must be used for each Section of the paper.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper. Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

SECTION A – Answer SIX parts of this section

1.1) Because of the centrifugal force which results from the rotation of the Earth, sea-level at the Poles is some tens of kilometres lower than that at the Equator. Explain why general relativity nevertheless requires that all clocks at sea-level run mutually in time.

[7 marks]

1.2) Describe the structure of a simple accelerometer.

Two spaceships travel in free fall on approximately parallel tracks. The accelerometers carried at the centre-of-mass of each spaceship both register zero. Describe a situation where nevertheless the velocity of one spaceship with respect to the other is not constant.

[7 marks]

1.3(i) A spaceship in remote space undergoes a steady acceleration g, as shown by its on-board accelerometer. Give a reason why a clock at the stern runs slow compared to a similar clock at the nose.

(ii) The on-board accelerometer of a spaceship at rest on its launching pad registers a steady acceleration g, on account of the presence of the Earth. Give a reason why a clock at the stern runs slow compared to a similar clock at the nose.

(iii) Is the Principle of Equivalence relevant to problems (i) and (ii)?

[7 marks]

1.4) An aircraft flies along a straight horizontal route over the North Pole. Describe the effects of (i) altitude and (ii) speed on the timekeeping of an accurate clock on board, as compared with a similar clock on the ground.

Determine whether a typical aircraft can fly so that the two clocks are synchronous. [Typical speeds and altitudes may be taken as, respectively, between 100 ms^{-1} and 300 ms^{-1}, and up to 10^4 m.]

[7 marks]

1.5) Determine the eight Christoffel symbols for the metric

$$\mathrm{d}s^2 = \mathrm{d}u^2 + \mathrm{d}\phi^2 \exp(2u).$$

[7 marks]

1.6) A certain surface is governed by the metric

$$ds^2 = du^2 + \frac{dv^2}{u},$$

in which the coordinate u is always positive.

Obtain two first integrals that are satisfied by u and v as functions of s along any geodesic.

Check that your result is satisfied by the known instance of a geodesic defined parametrically by

$$u = s - s^2 \qquad \text{and} \qquad v = s^2 - \frac{2}{3}s^3$$

in which $0 < s < 1$.

[7 marks]

1.7) One form of the metric for the surface of a sphere of radius R is

$$ds^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Three of the Christoffel symbols are nonzero; they are

$$\Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta \qquad \text{and} \qquad \Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \cot\theta.$$

Evaluate the 16 elements of the Riemann tensor, and show that the principal curvature of the surface is $+1/R^2$.

[7 marks]

1.8) In the Schwarzschild metric, the shape of an orbit associated with a geodesic worldline in the equatorial plane ($\theta = \pi/2$) is governed by

$$\left(\frac{du}{d\phi}\right)^2 = au^3 - u^2 + \alpha u + \beta,$$

where α and β are constants of the motion, and $u = 1/r$. *Assume* that a legitimate instance of this requirement is

$$\left(\frac{du}{d\phi}\right)^2 = a(u - u_0)^3,$$

where the constant $u_0 = 1/3a$. One solution is the circle $u = u_0$.

Show that a more general solution in this case is

$$u = u_0 + \frac{4}{a\phi^2},$$

and show that this solution yields a *spiral* orbit which tends towards the circle $r = 3a$ from the inside.

[7 marks]

SECTION B – Answer TWO questions

2(i) For each of the following three instances of phenomena related to the presence of the Sun, describe briefly and qualitatively the discrepancy between actual observation and the prediction of classical mechanics:

(a) the orbit of the planet Mercury;

(b) the line of sight to an object remote from the Solar System;

(c) the radar-location of an artificial satellite in orbit round the Sun.

[15 marks]

(ii) Choose one of these three instances (a), (b), or (c), and give a detailed description of the treatment of general relativity, showing in particular how a prediction better than that of classical mechanics is achieved.

[15 marks]

3) A spaceship travels in circular orbit round the Earth. The period of the ship in its orbit is $2\pi/\alpha$, where α is the constant angular frequency. For convenience on board, the crew use the slowly rotating coordinate system defined by:

Origin: centre-of-mass of the spaceship;
x-axis: tangent to the current direction of the orbit;
z-axis: radially away from the Earth centre;
y-axis: perpendicular to the other two axes;
t: time shown by a clock situated at the centre-of-mass.

Assume that for the crew on board the metric which includes the effects both of residual tidal forces and of centrifugal forces (to second order in the space coordinates) is:

$$c^2 \mathrm{d}\tau^2 = c^2 \left(1 + \frac{\alpha^2}{c^2}(y^2 - 3z^2)\right) \mathrm{d}t^2 - (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) + 2\alpha(x\mathrm{d}z - z\mathrm{d}x)\mathrm{d}t.$$

(i) Show that fixed clocks located wherever $y = \pm z\sqrt{3}$ keep strict step with the clock at the centre-of-mass.

[3 marks]

(ii) A particle in free fall inside the spaceship is seen to be moving along the y-axis. Show that its equation of motion $F_y = 0$ (when determined at first order) reduces to that for simple harmonic motion with period $2\pi/\alpha$. [Note that, at first order, $\dot{t}$ may be taken to be 1.]

[6 marks]

(iii) Observers on Earth see the orbit of the spaceship as circular. How do they see the orbit of the particle in (ii) above?

[4 marks]

(iv) A second particle is in free-fall motion in the xz-plane ($y = 0$). Obtain F_x and F_z, and hence show that the motion is governed by the simultaneous requirements

$$\ddot{x} + 2\alpha\dot{z} = 0,$$
$$\ddot{z} - 2\alpha\dot{x} - 3\alpha^2 z = 0.$$

Verify that one possible solution is

$$x = 2A\cos\alpha\tau, \qquad z = A\sin\alpha\tau.$$

Thus show that here the motion is round an ellipse with centre at the centre-of-mass.

[11 marks]

(v) How do observers on Earth see the orbit of the particle in (iv) above?

[6 marks]

4) The metric for a certain space of two dimensions with the coordinates x and t is

$$c^2 \mathrm{d}\tau^2 = c^2 \left(1 + \frac{2gx}{c^2}\right) \mathrm{d}t^2 - \frac{\mathrm{d}x^2}{1 + \dfrac{2gx}{c^2}}.$$

Here g is a constant, with dimensions of an acceleration.

(i) Write out the Lagrangian L for this metric, and derive the first integrals, valid for a geodesic worldline, namely:

$$c^2 \left(1 + \frac{2gx}{c^2}\right) \dot{t}^2 - \frac{\dot{x}^2}{1 + \dfrac{2gx}{c^2}} = c^2$$

$$\left(1 + \frac{2gx}{c^2}\right) \dot{t} = \text{constant.}$$

[8 marks]

(ii) Show that on any geodesic worldline the general solution for x is

$$x = x_0 - \frac{1}{2} g (\tau - \tau_0)^2$$

in which x_0 and τ_0 are disposable constants. Discuss the Newtonian limit ($c \to \infty, \dot{t} \to 1$).

[10 marks]

(iii) Note the facts that:

(a) as τ varies, x takes *all* values equal to or less than x_0;

(b) the original formula for $c^2 \mathrm{d}\tau^2$ becomes meaningless for values of x less than $-c^2/2g$.

How are (a) and (b) to be reconciled? In particular, is it correct to imply that x loses its meaning outside of a certain range? Relate your answer to the features known as *event horizons* and *black holes.*

[12 marks]

5) The metric for a uniform cosmology is

$$d\tau^2 = dt^2 - [R(t)]^2 d\Omega^2$$

in which $d\Omega^2$ is the metric for a three-dimensional space which is everywhere homogeneous and isotropic (with scalar curvature k), and the scale-factor $R(t)$ is a function of the epoch t. *Assume* that the energy-density ρ and pressure p are given by

$$8\pi\rho(t) = \frac{3(R'^2 + k)}{R^2},$$
$$8\pi p(t) = -\frac{(R'^2 + k)}{R^2} - \frac{2R''}{R}.$$

[Here, $R' \equiv dR/dt$ and $R'' \equiv d^2R/dt^2$. Throughout, units are chosen for which the speed of light and the universal gravitational constant are both unity.]

(i) Prove the identity

$$\rho' + (\rho + p)\frac{3R'}{R} \equiv 0.$$

Use this relation to show that the equation of state for the content of the universe is required to be adiabatic.

[11 marks]

(ii) The equation of state for a radiation-filled universe is $\rho = 3p$. Show that in this case ρR^4 is constant as epoch passes.

[3 marks]

(iii) Use the Stefan Law $\rho = \sigma T^4$ to discuss the behaviour of the background black-body radiation of our universe as epoch passes.

[4 marks]

(iv) Show that for a radiation-filled universe

$$R'^2 + k = \frac{\text{const}}{R^2}.$$

[4 marks]

(v) Given that the universe is currently expanding, explain how this differential equation for R suggests a finite history of the universe, beginning with a Big Bang.

[8 marks]