King's College London

UNIVERSITY OF LONDON

This paper is part of an examination of the College counting towards the award of a degree. Examinations are governed by the College Regulations under the authority of the Academic Board.

B.Sc. EXAMINATION

CP/3630 GENERAL RELATIVITY AND COSMOLOGY

Summer 1998

Time allowed: THREE Hours

Candidates should answer SIX parts of SECTION A, and TWO questions from SECTION B.

Separate answer books must be used for each Section of the paper.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

SECTION A – Answer SIX parts of this section

1.1) Describe the structure of a simple accelerometer.

Two spaceships travel in free fall in approximately parallel tracks. The accelerometers carried by each spaceship all register zero.

Describe a situation where nevertheless the velocity of one spaceship with respect to the other is not constant.

[7 marks]

1.2) In the neighbourhood of the Earth (Schwarzschild radius $= a$) we have the following two facts:

(i) an artificial satellite in a circular Earth orbit of radius r moves with speed v, where $v^2 = ac^2/2r$,

(ii) a clock at distance r moving with velocity v runs slow compared to an infinitely remote stationary clock by the factor

$$\sqrt{1 - \frac{v^2}{c^2} - \frac{a}{r}}.$$

Deduce from these facts that a clock in free circular orbit of radius $\frac{3}{2}\times$(Earth radius) will keep step with a similar clock fixed at the North Pole of the Earth.

[7 marks]

1.3) Two initially synchronised atomic clocks mounted in conventional aircraft are each flown once round the equator of the Earth, in opposite directions. They meet again after landing at their original take-off point. Which clock now shows the *later* time? Give a reason for your answer.

[7 marks]

1.4) The spacetime metric in the neighbourhood of the Earth is

$$c^2\mathrm{d}\tau^2 = c^2(1-a/r)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1-a/r} - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}\phi^2,$$

where a is the Schwarzschild radius and r, θ, ϕ are spherical polar coordinates. Two events (t, r_1, θ, ϕ) and (t, r_2, θ, ϕ) occur simultaneously, and in line with the centre of the Earth. Show that the actual radial distance D between the events is

$$D = \int_{r_1}^{r_2} \frac{\mathrm{d}r}{\sqrt{1-a/r}}.$$

Verify the approximation

$$\int \frac{\mathrm{d}r}{\sqrt{1-a/r}} = r + \frac{1}{2}a\ln r + O(a^2)$$

and use it to show that within 10^6 km above the surface of the Earth the discrepancy between D and $r_2 - r_1$ is at most a few centimetres. (Approximately, radius of the Earth $= 7\times10^3$ km, $a = 1$ cm.)

[7 marks]

1.5) The metric for plane polar coordinates (r, θ) is $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2$. *Assume* that any geodesic in the plane must satisfy both

$$\dot{r}^2 + r^2\dot{\theta}^2 = 1 \quad \text{and} \quad r^2\dot{\theta} = a,$$

where the value of the constant a depends on the geodesic.

Verify that these requirements are satisfied by

$$r(s) = \sqrt{s^2+a^2},$$
$$\theta(s) = \theta_0 + \arctan\frac{s}{a}.$$

Use a sketch to interpret the meanings of the constants a and θ_0.

[7 marks]

1.6) Explain how an inhomogeneous gravitational field manifests itself as a tidal force, affecting even a body in free fall.

A hollow sphere containing dust, but otherwise empty, is in free fall in a circular Earth orbit. Where inside the sphere will the dust tend to collect? Give reasons for your answer.

[7 marks]

1.7) A small clock, displaying proper time τ and originally at rest in the infinite past, is now falling radially through the Schwarzschild metric of a black hole of radius $a = 1$ cm. *Assume* that its motion is described by

$$r = ax^2, \qquad \theta = \text{const}, \qquad \phi = \text{const},$$
$$c\tau = \frac{2}{3}ax^3, \qquad ct = a\left(\frac{2}{3}x^3 + 2x + \ln\frac{x-1}{x+1}\right).$$

The parameter x increases from $-\infty$ through negative values. The other symbols have their usual meanings.

Show that a remote observer's last glimpse of the clock corresponds to $x = -1$, and that the clock itself survives for a further proper time of 2.2×10^{-11} s. (Ignore the catastrophic effect of tidal forces.)

[7 marks]

1.8) The Robertson–Walker metric for a smooth uniform universe is

$$\mathrm{d}\tau^2 = \mathrm{d}t^2 - [R(t)]^2\mathrm{d}S^2,$$

in which $\mathrm{d}S^2$ is the metric of a homogeneous 3D curved space. The curvature of the 3D space may be any of positive, zero, or negative; $R(t)$ is the scale at epoch t. *Assume* that the matter density in such a universe is

$$\rho = \frac{3}{8\pi R^2}\left[\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 + k\right],$$

where k is a constant which may have either sign.

If we are dealing with a dust-filled universe, for which $\rho R^3 = $ constant, show that $R(t)$ satisfies

$$\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 = \frac{A - kR}{R},$$

in which A is a positive constant.

Show qualitatively that if $\mathrm{d}R/\mathrm{d}t$ is currently greater than zero, then $R(t)$ must have been zero at some t in the past.

[7 marks]

SECTION B – Answer TWO questions

2) The Schwarzschild metric in the neighbourhood of a spherical mass is, in the usual notation,

$$c^2\mathrm{d}\tau^2 = c^2(1-a/r)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1-a/r} - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2).$$

(i) Using the general relation

$$2F_\mu = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu},$$

find F_t, F_r, F_θ, and F_ϕ. Hence show that one *contravariant* component of F is

$$F^r = \ddot{r} + \frac{ac^2(r-a)}{2r^3}\dot{t}^2 - \frac{a}{2r(r-a)}\dot{r}^2 - (r-a)\dot{\theta}^2 - (r-a)\sin^2\theta\dot{\phi}^2.$$

[14 marks]

(ii) Show that for an equatorial circular orbit (with $\theta = \pi/2$) centred on the spherical mass, F^r reduces to

$$F^r = (r-a)\dot{t}^2\left[\frac{ac^2}{2r^3} - \left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2\right].$$

[5 marks]

(iii) Show that for a free-fall circular orbit (as followed for example by a typical Earth satellite) the third Kepler Law is satisfied.

[4 marks]

(iv) For a laboratory built at the equator of the Earth, show that to good approximation

$$F^r = \frac{ac^2}{2r^2} - r\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2.$$

[4 marks]

(v) Interpret each term in this equation.

[3 marks]

3) The differential equation which governs a planetary orbit $1/r = u(\phi)$ in a Schwarzschild metric of Schwarzschild radius a is

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 = (u - u_1)(u_2 - u)(1 - au_1 - au_2 - au).$$

Here the constants u_1 and u_2 are respectively the values of $1/r$ at aphelion and perihelion.

(i) Show that the substitution

$$u = \frac{u_1 + u_2 q^2}{1 + q^2}$$

in the differential equation results in

$$\frac{\mathrm{d}q}{\mathrm{d}\phi} = \frac{1}{2}(1 + q^2)\sqrt{1 - au_1 - au_2 - a\frac{u_1 + u_2 q^2}{1 + q^2}}.$$

Deduce that the change in ϕ between consecutive perihelia is

$$\Delta\phi = \int_{-\infty}^{\infty} \frac{2\mathrm{d}q}{(1 + q^2)\sqrt{1 - au_1 - au_2 - a\frac{u_1 + u_2 q^2}{1 + q^2}}}.$$

[14 marks]

(ii) For a large enough *circular* orbit, show that $\Delta\phi = 2\pi/\sqrt{(1 - 3au_1)}$. Hence deduce the approximate perihelion precession per circuit for a very nearly circular orbit.

[11 marks]

(iii) Discuss qualitatively what happens for a very nearly circular orbit when $1/u_1 \leq 3a$.

[5 marks]

Note: You may assume that $\int_{-\infty}^{\infty} \frac{\mathrm{d}q}{1+q^2} = \pi$.

4) A model universe is described by

$$d\tau^2 = dt^2 - \alpha t^{2n} d\Omega^2$$

where α and n are constants, and $d\Omega^2$ is the metric for a homogeneous isotropic curved three-dimensional space. In all that follows, differentiation with respect to t is denoted by a *dash*: f' denotes $df(t)/dt$, and so on.

(i) Evaluate the energy density ρ and the pressure p as functions of the epoch t and verify that they are related by the adiabatic requirement

$$\rho' + \frac{3n}{t}(\rho + p) = 0.$$

[9 marks]

(ii) Relate this result to the conservation of energy during the adiabatic expansion of a gas enclosed in a container with a variable volume V.

Discuss the validity of conservation of energy for the universe as a whole.

[9 marks]

(iii) For the case when the space-metric $d\Omega^2$ is flat, show that the equation of state which relates ρ and p simplifies to

$$p = \rho\left(\frac{2}{3n} - 1\right).$$

[5 marks]

(iv) By considering the instances a) dust, and b) pure radiation, show that if the metric with flat $d\Omega^2$ is to be used as a model of a universe with conventional behaviour, then

$$\frac{1}{2} \leq n \leq \frac{2}{3}.$$

[7 marks]

Note: For the metric

$$d\tau^2 = dt^2 - [R(t)]^2 d\Omega^2$$

you may assume that $\rho(t)$ and $p(t)$ are related to the scale-factor R by

$$8\pi\rho = \frac{3(R'^2 + k)}{R^2} \qquad \text{and} \qquad 8\pi p = -\frac{R'^2 + k}{R^2} - \frac{2R''}{R},$$

in which k is the curvature of the homogeneous isotropic three-dimensional space with metric $d\Omega^2$.

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5) A surface of infinite extent is described by a pair of coordinates u and v, in the ranges $0 < u < \infty$ and $-\infty < v < \infty$. The metric for the surface is

$$\mathrm{d}s^2 = \frac{\mathrm{d}u^2 + \mathrm{d}v^2}{u^2}.$$

(i) Obtain two first integrals, relating u, $\mathrm{d}u/\mathrm{d}s$, and $\mathrm{d}v/\mathrm{d}s$, to be satisfied by any geodesic. Deduce that for every geodesic u and $\mathrm{d}u/\mathrm{d}s$ must satisfy a relation

$$\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2 = u^2 - Au^4,$$

in which A is a non-negative constant.

[6 marks]

(ii) Verify that, when $A = 0$, the most general solution is

$$u = \mathrm{e}^{\pm(s - s_0)}, \qquad v = v_0,$$

in which s_0 and v_0 are constants of integration. Explain why it is sufficient to adopt

$$u = \mathrm{e}^{s}, \qquad v = v_0$$

as a general solution.

[6 marks]

(iii) Verify that for $A > 0$, a general solution is

$$u = \frac{C}{\cosh s}, \qquad v = v_0 + \frac{C \sinh s}{\cosh s},$$

provided that $C > 0$ and $AC^2 = 1$.

[10 marks]

(iv) Show that in (iii) $u^2 + (v - v_0)^2 =$ constant. Hence show that in a map for which u and v are Cartesian coordinates the geodesics are represented as semicircles.

[4 marks]

(v) In the (u, v) diagram, sketch a few examples of the geodesics in each of (ii) and (iii) above.

[4 marks]