# King's College London

UNIVERSITY OF LONDON



B.Sc. EXAMINATION

CP/3630 GENERAL RELATIVITY AND COSMOLOGY

Summer 1997

Time allowed: THREE Hours

**Candidates should answer SIX parts of SECTION A, and TWO questions from SECTION B.**

**Separate answer books must be used for each Section of the paper.**

**The approximate mark for each part of a question is indicated in square brackets.**

**You must not use your own calculator for this paper. Where necessary, a College calculator will have been supplied.**

TURN OVER WHEN INSTRUCTED

## SECTION A – Answer SIX parts of this section

1.1) An astronaut travels in a spaceship without windows or other contact with the outside world. Can the acceleration of the spaceship be measured? If the answer to this question is 'yes', how may it be done? What will the accelerometer register in free orbit round the Earth?

[7 marks]

1.2) Two aircraft fly at the same constant speed and at the same constant altitude along the equator of the Earth, one easterly and the other westerly. As they pass, their clocks are synchronised. When next they pass (180° later), the clocks are compared. Explain why the eastward–travelling clock is found to have run *slow* compared to the westward one. (Hint: The Earth is rotating.)

[7 marks]

1.3) A spaceship in deep space is being uniformly accelerated by its rocket motor. Two clocks are carried, one at the nose, the other at the tail. Explain why these clocks do not keep in step. Which clock runs faster?

[7 marks]

1.4) The spacetime metric in the neighbourhood of the Earth is

$$c^2 \mathrm{d}\tau^2 = c^2 (1 - a/r)\, \mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - a/r} - r^2 \mathrm{d}\theta^2 - r^2 \sin^2\theta \, \mathrm{d}\phi^2,$$

where $a$ is the Schwarzschild radius, and $r, \theta, \phi$ are spherical polar coordinates. A pulse of light is emitted (at $t_0, r_0, \theta_0, \phi_0$) and travels radially outward. Show that subsequently the coordinates $r$ and $t$ of the pulse are related by

$$\frac{\mathrm{d}r}{\mathrm{d}t} = c\left(1 - \frac{a}{r}\right).$$

By differentiation or otherwise, verify that

$$c(t - t_0) = r - r_0 + a \ln \frac{r - a}{r_0 - a}.$$

Show that within $10^6$ km above the surface of the Earth the discrepancy between this and the Newtonian result is at most several centimetres. (Approximately, radius of the Earth = 7000 km, $a = 1$ cm.)

[7 marks]

1.5) The metric for a certain surface with coordinates $(r, \theta)$ is

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{r^2} + r^2 \mathrm{d}\theta^2.$$

*Assume* that any geodesic on the surface must satisfy both

$$\frac{\dot{r}^2}{r^2} + r^2\dot{\theta}^2 = 1 \quad \text{and} \quad r^2\dot{\theta} = a,$$

where the value of the constant $a$ depends on the geodesic.

Verify that these requirements are satisfied by

$$r(s) = a\cosh s,$$
$$\theta(s) = \theta_0 + \frac{\sinh s}{a\cosh s}.$$

Use a sketch to interpret the meanings of the constants $a$ and $\theta_0$.

[7 marks]

1.6) A hollow sphere containing dust is in orbit round the Earth. Where inside the sphere will the dust tend to collect?

Relate your answer to the tidal effect of the Moon on the oceans of the Earth.

[7 marks]

1.7) A particle is in circular orbit in the Schwarzschild metric of a black hole of radius $a$. *Assume* that its motion is described by

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - 3a/2r}},$$
$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \sqrt{\frac{\frac{1}{2}ac^2}{r^3(1 - 3a/2r)}},$$

where the symbols have their usual meanings.

What is the orbit of minimum possible size? What kind of particle is able to follow this orbit?

[7 marks]

1.8) The Robertson–Walker metric for a smooth uniform universe is

$$\mathrm{d}\tau^2 = \mathrm{d}t^2 - [R(t)]^2 \mathrm{d}S^2,$$

in which $\mathrm{d}S^2$ is the metric of a homogeneous three–dimensional curved space, and $R(t)$ is the scale at epoch $t$. The curvature $k$ of the three-dimensional space may be positive, zero, or negative. *Assume* that the matter density in such a universe is

$$\rho = \frac{3}{8\pi R^2}\left[\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 + k\right].$$

If we are dealing with a radiation–filled universe, for which $\rho R^4$ = constant, show that $R(t)$ satisfies

$$\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 = \frac{A^2 - kR^2}{R^2},$$

in which $A$ is a constant.

Show qualitatively that if $\mathrm{d}R/\mathrm{d}t$ is currently greater than zero, then $R(t)$ must have been zero at some epoch $t$ in the past.

[7 marks]

## SECTION B – Answer TWO questions

2) Spacetime in the vicinity of the Earth is governed by the usual Schwarzschild metric

$$c^2 \mathrm{d}\tau^2 = c^2(1 - a/r)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - a/r} - r^2 \mathrm{d}\theta^2 - r^2 \sin^2\theta \, \mathrm{d}\phi^2$$

where the Schwarzschild radius $a = 8.8 \times 10^{-3}\,\mathrm{m}$, and the speed of light $c = 3 \times 10^8\,\mathrm{m\,s^{-1}}$.

(i) A clock follows an arbitrary track on the surface of the sphere $r = r_0$. Show that the proper time $\tau$ of the clock is related to the coordinate time $t$ by

$$c^2 \left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 = c^2 \left(1 - \frac{a}{r_0}\right) - v^2,$$

in which

$$v^2 = r_0^2 \left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 + r_0^2 \sin^2\theta \left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2.$$

[8 marks]

(ii) Show that $v$ is the magnitude of the velocity of the clock as measured by a remote observer at rest.

[8 marks]

(iii) Clock A is fixed at ground level on the equator of the Earth, and clock B is immediately overhead in an aircraft flying due North with speed $V$ and at altitude $H$. Clocks A and B are observed to be strictly in step.

Show that

$$\frac{ac^2}{R} + V_E^2 = \frac{ac^2}{R + H} + (V_E^2 + V^2),$$

where $R = 6.4 \times 10^6\,\mathrm{m}$ is the radius of the equator, and $V_E$ is the eastward velocity of either clock which results from the rotation of the Earth.

[8 marks]

(iv) By considering a typical aircraft speed (say $V = 200\,\mathrm{m\,s^{-1}}$), show that the situation of (iii) is easily achieved.

[6 marks]

3) A narrow light beam from infinity is deflected by the Schwarzschild metric (with coordinates $t, r, \theta, \phi$) associated with the exterior of a spherically symmetric mass. *Assume* that the beam is entirely in the equatorial plane $\theta = \pi/2$, and that its orbit satisfies

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 = au^3 - u^2 + \frac{1}{b^2}$$

in which $u = 1/r$, $a$ is the Schwarzschild radius for the mass, and $b$ is the impact parameter of the beam. For the Sun, $a = 2.96\,\mathrm{km}$.

(i) If the beam passes through a perihelion, show that

$$b = R\sqrt{\frac{R}{R-a}} = R + \frac{a}{2} + O(a^2)$$

where $R$ is the value of $r$ at the perihelion.

[8 marks]

(ii) Deduce from (i) that the apparent diameter of the Sun as seen by a remote observer is larger than the true diameter by about 3 km. Why is this prediction difficult to observe directly?

[6 marks]

(iii) If the mass is a black hole of 'radius' $a$, show that the minimum value of $b$ for real $R$ in the range $a < R < \infty$ is $a\sqrt{27/4}$, reached at $R = 3a/2$.

[8 marks]

Discuss the behaviour of a beam with an impact parameter $b < a\sqrt{27/4}$, and deduce the apparent diameter of the black hole as seen by a distant observer.

[8 marks]

4) The Robertson–Walker metric for a smooth uniform universe is

$$d\tau^2 = dt^2 - [R(t)]^2 dS^2$$

in which $dS^2$ is the metric of a homogeneous three-dimensional curved space of constant curvature $k$, and $R(t)$ is the scale at epoch $t$. *Assume* that the general formulae for mass–density $\rho$ and pressure $p$ in the Robertson–Walker context are

$$\rho = \frac{3}{8\pi R^2}\left[\left(\frac{dR}{dt}\right)^2 + k\right],$$

$$p = -\frac{1}{8\pi R^2}\left[\left(\frac{dR}{dt}\right)^2 + k\right] - \frac{1}{4\pi R}\frac{d^2R}{dt^2}.$$

(i) The scale $R(t)$ of a certain universe U is governed by the requirement

$$\left(\frac{dR}{dt}\right)^2 = \frac{A}{R} - k$$

in which $A > 0$ is a constant. Show that, for the universe U,

$$\rho R^3 = \text{constant} \qquad \text{and} \qquad p = 0.$$

[10 marks]

(ii) Explain why the universe U is described as *dust–filled.*

[5 marks]

(iii) The function $q(t)$ is related to $R(t)$ by

$$R = \frac{A}{q^2 + k}.$$

Show that, for the universe U,

$$2A\frac{dq}{dt} = (q^2 + k)^2,$$

apart from an unimportant ambiguity in sign. For the case $k = 1$, verify that $q$ and $t$ are related by

$$t = A\left(\frac{q}{q^2 + 1} + \arctan q\right) + \text{constant}.$$

[10 marks]

(iv) Sketch a plot of $R$ against $t$ as the parameter $q$ goes from $-\infty$ to $+\infty$. What is the lifetime of the universe U?

[5 marks]

5) A certain two–dimensional space is described by a pair of coordinates $u$ and $v$, the metric being

$$\mathrm{d}s^2 = \mathrm{e}^{u^2+v^2}(\mathrm{d}u^2 - \mathrm{d}v^2).$$

(i) Show that the Lagrangian for this metric is

$$L = \mathrm{e}^{u^2+v^2}(\dot{u}^2 - \dot{v}^2).$$

in which a *dot* signifies a derivative with respect to $s$.

[4 marks]

(ii) Use the general prescription

$$2F_\mu = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu}$$

to obtain

$$\begin{aligned} F^u &= \ddot{u} + u\dot{u}^2 + 2v\dot{u}\dot{v} + u\dot{v}^2, \\ F^v &= \ddot{v} + v\dot{u}^2 + 2u\dot{u}\dot{v} + v\dot{v}^2. \end{aligned}$$

[8 marks]

(iii) List all eight Christoffel symbols, and assemble them into a pair of $(2 \times 2)$–matrices $\mathbf{\Gamma}_u$, $\mathbf{\Gamma}_v$ in the standard way.

[9 marks]

(iv) Evaluate

$$\mathbf{B}_{uv} = \frac{\partial \mathbf{\Gamma}_v}{\partial u} - \frac{\partial \mathbf{\Gamma}_u}{\partial v} + \mathbf{\Gamma}_u \mathbf{\Gamma}_v - \mathbf{\Gamma}_v \mathbf{\Gamma}_u,$$

and hence show that the space is flat.

[9 marks]