King's College London

UNIVERSITY OF LONDON

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B.Sc. EXAMINATION

CP3630 General Relativity and Cosmology

Summer 2005

Time Allowed: THREE Hours

Candidates should answer all SIX parts of SECTION A, and no more than TWO questions from SECTION B. No credit will be given for answering further questions.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

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Physical Constants

Permittivity of free space	$\epsilon_0 = 8.854 \times 10^{-12}$ F m^{-1}
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}$ H m^{-1}
Speed of light in free space	$c = 2.998 \times 10^{8}$ m s^{-1}
Gravitational constant	$G = 6.673 \times 10^{-11}$ N m^2 kg^{-1}
Elementary charge	$e = 1.602 \times 10^{-19}$ C
Electron rest mass	$m_{\mathrm{e}} = 9.109 \times 10^{-31}$ kg
Unified atomic mass unit	$m_{\mathrm{u}} = 1.661 \times 10^{-27}$ kg
Proton rest mass	$m_{\mathrm{p}} = 1.673 \times 10^{-27}$ kg
Neutron rest mass	$m_{\mathrm{n}} = 1.675 \times 10^{-27}$ kg
Planck constant	$h = 6.626 \times 10^{-34}$ J s
Boltzmann constant	$k_{\mathrm{B}} = 1.381 \times 10^{-23}$ J K^{-1}
Stefan-Boltzmann constant	$\sigma = 5.670 \times 10^{-8}$ W m^{-2} K^{-4}
Gas constant	$R = 8.314$ J mol^{-1} K^{-1}
Avogadro constant	$N_{\mathrm{A}} = 6.022 \times 10^{23}$ mol^{-1}
Molar volume of ideal gas at STP	$= 2.241 \times 10^{-2}$ m^3
One standard atmosphere	$P_0 = 1.013 \times 10^{5}$ N m^{-2}
Schwarzschild metric (SM) (in units with $G = c = 1$)	$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$
Shell coordinates in SM:	$dt_{\mathrm{shell}} = (1 - \frac{2M}{r})^{1/2} dt$; $dr_{\mathrm{shell}} = (1 - \frac{2M}{r})^{-1/2} dr$
Christoffel symbols:	$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta})$
Riemann Curvature Tensor (RCT):	$R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}{}_{\beta\nu,\mu} - \Gamma^{\alpha}{}_{\beta\mu,\nu} + \Gamma^{\alpha}{}_{\kappa\mu}\Gamma^{\kappa}{}_{\beta\nu} - \Gamma^{\alpha}{}_{\kappa\nu}\Gamma^{\kappa}{}_{\beta\mu}$
Properties of RCT:	$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}$
Ricci tensor:	$R_{\mu\nu} = R_{\nu\mu} = R^{\alpha}{}_{\mu\alpha\nu}$
Cosmic Horizon in Friedmann-Robertson-Walker Universe:	$\delta(t_0) = a(t_0) \int_{t_0}^{\infty} \frac{dt'}{a(t')}$, (in units with $c = 1$)

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SECTION A - Answer all SIX parts of this section

1.1) Consider the two-dimensional metric space $ds^2 = dr^2 - r^2 du^2$, where (r, u) are some coordinates. Without calculating the Christoffel symbols, but by invoking the coordinate transformation $(r, u) \to (x, t)$, where $x = r \cosh u$, $t = r \sinh u$, show that the geodesics of this spacetime are straight lines.

[7 marks]

1.2) Consider a D-dimensional space-time whose Ricci tensor has the form $R_{\mu\nu} = A g_{\mu\nu}$, where the constant $A > 0$, and $g_{\mu\nu}$ is the metric tensor. Show that this space time is an exact solution of Einstein equations without matter but with a cosmological constant, and determine this constant in the case of four space-time dimensions.

[7 marks]

1.3) Using the red-shift formula, explain qualitatively what happens to the wavelength of a photon emitted radially at time t_1 and received at time t_2 by observers at rest with respect to an expanding (flat) Robertson–Walker spacetime described by the metric

$$ds^2 = -dt^2 + a^2(t)\big\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\big\}.$$

[7 marks]

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1.4) Consider a spherically-symmetric non-rotating body of mass M. Let two concentric shells surrounding the body be located at $r_1 = 4M$ and $r_2 = 8M$, where r denotes the radial spherical polar coordinate, and work in a system of units in which the mass is measured in units of length. Show that the period of this light is increased by a factor 1.22 when it is emitted from the shell r_1 and absorbed at the shell r_2.

[7 marks]

1.5) Starting from rest at a great distance an observer is plunging straight (i.e. radially) towards a non-rotating black hole of mass equal to eight solar masses $8M_\odot$, where the mass of sun is $M_\odot = 1.989 \times 10^{30}$ kg or in geometrized units ($G = c = 1$) $M_\odot = 1.477 \times 10^3$ meters.

The observer sets his wristwatch to noon as he determines (by one means or another) that he is crossing the horizon. Determine how much time (in seconds) is left, according to the wristwatch of the observer, until the instant of crunch (i.e. when he reaches the singularity). Assume without proof the formula for the energy in the Schwarzschild geometry, involving proper (wristwatch) and far-away times.

[You may assume without proof that the energy E of a particle of mass m in the Schwarzschild geometry is given by: $\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$, where τ is the proper time.]

[7 marks]

1.6) Assume that, under general coordinate transformations $x^\mu \to x'^\mu(x^\nu)$, a co-vector $\mathcal{V}_\mu$ and a covariant second rank tensor $\mathcal{T}_{\mu\nu}$ transform as follows: $\mathcal{V}_\mu \to \mathcal{V}\,'_\mu = \frac{\partial x^\alpha}{\partial x'^\mu}\mathcal{V}_\alpha$, and $\mathcal{T}_{\mu\nu} \to \mathcal{T}\,'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\mathcal{T}_{\alpha\beta}$, respectively. Show that the object $\partial_\mu A_\nu$ is not a tensor under general coordinate transformations, where ∂_μ is the ordinary partial derivative with respect to x^μ and A_ν are the components of a covector.

[7 marks]

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SECTION B - Answer TWO questions

2) Consider the two-dimensional spacetime described by the infinitesimal line element:

$$ds^2 = -dt^2 + e^{2Ht} dr^2,$$

where t is the time coordinate and $H > 0$ is a constant.

(a) What do you conclude on the existence of a cosmic horizon in this geometry?

[3 marks]

(b) By using an appropriate variational method, or otherwise, compute the Christoffel symbols for the above spacetime.

[6 marks]

(c) Compute the independent components of the Riemann tensor in this two dimensional geometry.

[6 marks]

(d) Compute the non-vanishing components of the Ricci tensor for this spacetime, and show that they satisfy the relation:

$$R_{\mu\nu} = H^2 g_{\mu\nu}$$

[8 marks]

(e) Compute the curvature scalar of this spacetime, and discuss the evolution in cosmic time. Discuss the behaviour of the universe in the two limiting cases $t \to \infty$ and $t \to -\infty$.

[7 marks]

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3) (a) Define what one means by the term "a null curve" in a generic D-dimensional space-time with coordinates x^α, $\alpha = 0, 1, \ldots D-1$, and metric $g_{\mu\nu}$. A conformal transformation of a metric $g_{\mu\nu}$ is defined by $g_{\mu\nu} \to f(x) g_{\mu\nu}$, where $f(x)$ is a positive-definite, real, but otherwise arbitrary, function of the coordinates x^α. Show that all null curves remain null under this transformation.

[5 marks]

(b) Consider the three-dimensional space time:

$$ds^2 = e^{2\phi(x)} \eta_{\alpha\beta} dx^\alpha dx^\beta \ , \qquad \alpha, \beta = 0, 1, 2$$

where $\eta_{\alpha\beta}$ denotes the three-dimensional Minkowski metric, and $\phi(x)$ is a real scalar function of the coordinates x^α. Show that the appropriate *null* geodesic equation reads:

$$\frac{d^2 x^\alpha}{d\lambda^2} + 2 \frac{dx^\alpha}{d\lambda} \frac{dx^\mu}{d\lambda} \, \partial_\mu \phi \ = 0 \ ,$$

where λ is the affine parameter parametrizing the geodesic.

[20 marks]

(c) By multiplying the null geodesic equation of (b) by $e^{2\phi}$, and rescaling λ such that $d\lambda' = e^{-2\phi} d\lambda$, show that

$$\frac{d^2 x^\alpha}{d\lambda'^2} = 0$$

and thus determine the shape of the null geodesics of (b). Explain your result based on the result in (a) above.

[Hint: Use the identity $\frac{d\phi}{d\lambda} = \frac{dx^\mu}{d\lambda} \, \partial_\mu \phi$ to rewrite the null geodesic of part (b) in terms of $d\phi/d\lambda$]

[5 marks]

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4) **(a)** For an expanding Universe with scale factor $a(t)$, regarded as an ideal fluid, the change in the total energy dE satisfies the thermodynamic relation: $dE = -p\ dV$, where dV denotes the change in the proper (spatial) volume, and p is the pressure. Show that for a fluid with equation of state $p = w\rho$, with w a constant, and ρ the mass density, one obtains:

$$a\frac{d\rho}{da} + 3(1+w)\rho = 0 \tag{1}$$

[8 marks]

(b) Integrate Eq. (1) to obtain the scaling law of ρ as a function of $a(t)$. State the value of w which characterises the radiation dominated era of the Universe and, hence, give the scaling law of ρ versus $a(t)$ for that era.

[8 marks]

(c) Using the Stefan-Boltzmann law for the energy of thermal radiation, $E_{\rm rad} = \alpha\, T^4$, where α is the radiation constant, show that the temperature of a radiation-dominated Universe is inversely proportional to the scale factor.

[7 marks]

(d) Using the expression for the cosmological redshift, which you should state without proof, show that the result of (c) above implies Wien's law of thermodynamics, according to which the maximum of a thermal radiation spectrum has a wavelength $\lambda_{\rm max}$ which changes with the temperature $T_{\rm rad}$ of radiation according to: $\lambda_{\rm max}\, T_{\rm rad} = $ constant.

[7 marks]

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