King's College London

UNIVERSITY OF LONDON

This paper is part of an examination of the University counting towards the award of a degree. Examinations of the University are governed by the Senate Regulations.

BSc Examination

CP/3630 GENERAL RELATIVITY AND COSMOLOGY

SUMMER 2000

TIME ALLOWED: THREE HOURS

Candidates must answer

SIX parts of **SECTION A**, and

TWO questions of **SECTION B**.

The provisional mark for each question or part of a question is indicated in square brackets.

Candidates are permitted to use calculators provided by the Registry only.

TURN OVER WHEN INSTRUCTED

SECTION A — Answer SIX parts of this section.

1.1) State the Principle of Equivalence.
A spaceship with engine running is poised, stationary, just above its launch pad on Earth. An identical spaceship with its engine running exactly as the first is in deep space. Among other things, the Principle of Equivalence claims that the relative difference in clock-rate at bow and stern is the same on either spaceship.
Justify this claim by relating the Pound-Rebka effect (in one case) to the Doppler shift (in the other case).

[7 marks]

1.2) Two similar aircraft are at the same airfield on the equator of the Earth. Each carries an accurate clock; these clocks are identical, and have been synchronised. Aircraft A then flies easterly once round the equator, at an agreed altitude, and lands again at the airfield. Aircraft B does the same, but in the westerly direction. When compared, the clock on aircraft B is now found to show a time *later* than shown by the clock on aircraft A.
Explain how this result may be predicted in the context of General Relativity.

[7 marks]

1.3) A hollow sphere containing dust is in orbit round the Earth. Where in the sphere will the dust tend to collect, and why?
Relate this effect to the tidal effect of the Moon upon the oceans of the Earth.

[7 marks]

1.4) On account of the centrifugal force associated with the rotation of the Earth, sea-level at the Equator is many kilometres higher than at either Pole. Explain why nevertheless all clocks at sea-level run mutually in time.

[7 marks]

1.5) A two-dimensional surface with coordinates x and y has the metric

$$\mathrm{d}s^2 = (\mathrm{d}x^2 + \mathrm{d}y^2)\exp 2x.$$

Verify that a geodesic on the surface may be described parametrically by functions $x(s)$ and $y(s)$ defined by:

$$x(s) = \tfrac{1}{2}\ln(s^2 + a^2), \quad \text{and}$$
$$y(s) = y_0 + \arctan\frac{s}{a},$$

in which a and y_0 are arbitrary constants.

[7 marks]

1.6) A two-dimensional surface with coordinates u and v has the metric

$$\mathrm{d}s^2 = (\mathrm{d}u^2 + \mathrm{d}v^2)\exp 2u.$$

Determine the eight Christoffel symbols.

[7 marks]

1.7) A two-dimensional 'spacetime' has the metric

$$c^2\mathrm{d}\tau^2 = c^2\left(1 + \frac{2gx}{c^2}\right)\mathrm{d}t^2 - \frac{\mathrm{d}x^2}{1 + \dfrac{2gx}{c^2}}$$

in which g is a constant with the dimensions of acceleration. *Assume* that the motion of a certain particle in free fall is specified parametrically by

$$x = -\tfrac{1}{2}g\tau^2, \qquad t = \frac{c}{2g}\ln\frac{c + g\tau}{c - g\tau}.$$

Examine the behaviour of x, of t, and of the metric, as τ goes from $-\infty$ to $+\infty$. Relate this behaviour to the concepts of *black hole* and *event horizon.*

[7 marks]

1.8) A two-dimensional surface with coordinates θ and ϕ has the metric

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sinh^2\theta\,\mathrm{d}\phi^2.$$

Assume that three of the Christoffel symbols are

$$\Gamma^{\theta}{}_{\phi\phi} = -\sinh\theta\cosh\theta, \qquad \Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \frac{\cosh\theta}{\sinh\theta},$$

the remaining five being zero.
Show that the curvature of the surface is -1 everywhere.

[7 marks]

SECTION B — Answer TWO questions.

2) A region of spacetime carries the coordinates t, x, y, z, and its metric is given by

$$c^2 \mathrm{d}\tau^2 = c^2 \left(1 + \frac{2\Phi}{c^2}\right) \mathrm{d}t^2 - \frac{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}{1 + \dfrac{2\Phi}{c^2}},$$

where the function $\Phi(x, y, z)$ is a scalar field. (Differentiation with respect to τ will be denoted by a dot. Also Φ_x will be written for $\partial\Phi/\partial x$, and similarly for Φ_y and Φ_z.)

(i) Obtain L (the Lagrangian) and $\partial L/\partial \dot{t}$, and show that, for any free-fall motion,

$$\frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \Phi = \text{constant}.$$

[9 marks]

(ii) Obtain the component F_x of the fourvector $\mathbf{F}$ by

$$2F_x = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x},$$

and show that

$$F^x = \ddot{x} - \frac{2\dot{x}(\dot{x}\Phi_x + \dot{y}\Phi_y + \dot{z}\Phi_z)}{c^2 + 2\Phi} + \left(1 + \frac{2\Phi}{c^2}\right)\dot{t}^2\Phi_x + \frac{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\Phi_x}{c^2 + 2\Phi}.$$

[9 marks]

(iii) Show that in the Newtonian limit of infinite speed of light ($c \to \infty$) and universal time ($\dot{t} \to 1$), F^x reduces to

$$F^x = \ddot{x} + \Phi_x.$$

[5 marks]

(iv) Use the results of (i) and (iii) above to interpret Φ as gravitational potential energy per unit mass in the Newtonian limit.

[7 marks]

3) The differential equation governing the shape of a general equatorial free-fall orbit in a Schwarzschild metric, with coordinates t, r, θ, ϕ, is

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 = au^3 - u^2 + \alpha u + \beta,$$

where $u \equiv 1/r$ and a is a constant. The size and shape of the orbit are related to the values of the parameters α and β.

(i) Verify that one instance of this differential equation is

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 = au\left(u - \frac{1}{2a}\right)^2.$$

[2 marks]

(ii) A new variable v is defined by $2au = v^2$, with $v > 0$ always. Show that the differential equation shown in (i) implies that v must satisfy

$$\frac{\mathrm{d}v}{\mathrm{d}\phi} = \pm\frac{1}{\sqrt{8}}(v^2 - 1).$$

[3 marks]

(iii) Describe the orbit which corresponds to the solution $v = 1$.

[2 marks]

(iv) Verify that a more general solution $v(\phi)$ of the differential equation in (ii) is defined by

$$\frac{v-1}{v+1} = A \exp\frac{\phi}{\sqrt{2}}.$$

in which A is a constant of integration.

[6 marks]

(v) If $A < 0$, show that $0 < v < 1$, and that the corresponding orbit takes a spiral shape with r in the range $2a < r < \infty$. Discuss the range of values of ϕ.

[8 marks]

(vi) Discuss similarly the case $A > 0$.

[9 marks]

4) The Robertson-Walker metric for a smooth uniform universe is

$$d\tau^2 = dt^2 - [R(t)]^2 \, dS^2$$

in which dS^2 is the metric of a homogeneous three-dimensional curved space of constant curvature k ($= 1, 0,$ or -1), and $R(t)$ is the scale at epoch t. *Assume* that the mass density and pressure in such a universe are given by

$$8\pi\rho = \frac{3}{R^2}\left[\left(\frac{dR}{dt}\right)^2 + k\right],$$
$$8\pi p = -\frac{1}{R^2}\left[\left(\frac{dR}{dt}\right)^2 + k\right] - \frac{2}{R}\frac{d^2R}{dt^2}.$$

(i) Verify that ρ, p, and R are related by

$$\frac{d\rho}{dt} + (\rho + p)\frac{3}{R}\frac{dR}{dt} = 0.$$

[6 marks]

(ii) Assume that for black-body radiation, $p = \frac{1}{3}\rho$. Show that ρR^4 is a constant for a universe filled with black-body radiation alone.

[5 marks]

(iii) For a universe in which ρR^4 is a constant, show that $R(t)$ satisfies the differential equation

$$\left(\frac{dR}{dt}\right)^2 = \frac{A^2}{R^2} - k$$

in which A is a constant.

[3 marks]

(iv) Verify that solutions $R(t)$ to this differential equation are given by

$$R^2 = \begin{cases} k(A^2 - t^2) & \text{if } k = \pm 1; \\ \pm 2At & \text{if } k = 0. \end{cases}$$

[7 marks]

(v) For each value of k, sketch graphs of R against t, wherever R is real and positive. By inspecting the sketches, or otherwise, show that if $R(t)$ is currently increasing as t passes, then a Big Bang must have occurred at some time in the past.

[9 marks]

5) A surface of infinite extent is described by a pair of coordinates u and v, in the ranges $0 < u < \infty$ and $-\infty < v < \infty$. The metric for the surface is

$$\mathrm{d}s^2 = \frac{\mathrm{d}u^2 + \mathrm{d}v^2}{u^2}.$$

(i) Obtain two first integrals, relating u, $\mathrm{d}u/\mathrm{d}s$, and $\mathrm{d}v/\mathrm{d}s$, to be satisfied by any geodesic. Deduce that for every geodesic u and $\mathrm{d}u/\mathrm{d}s$ must satisfy a relation

$$\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2 = u^2 - Au^4,$$

in which A is a non-negative constant.

[6 marks]

(ii) Verify that, when $A = 0$, the most general solution is

$$u = \mathrm{e}^{\pm(s - s_0)}, \qquad v = v_0,$$

in which s_0 and v_0 are constants of integration. Explain why it is sufficient to adopt

$$u = \mathrm{e}^{s}, \qquad v = v_0$$

as a general solution.

[6 marks]

(iii) Verify that for $A > 0$, a general solution is

$$u = \frac{C}{\cosh s}, \qquad v = v_0 + \frac{C \sinh s}{\cosh s},$$

provided that $C > 0$ and $AC^2 = 1$.

[10 marks]

(iv) Show that in (iii) $u^2 + (v - v_0)^2 = \text{constant}$. Hence show that in a map for which u and v are Cartesian coordinates the geodesics are represented as semicircles.

[4 marks]

(v) In the (u, v) diagram, sketch a few examples of the geodesics in each of (ii) and (iii) above.

[4 marks]

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