King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/3201 MATHEMATICAL METHODS IN PHYSICS III

Summer 1997

Time allowed: THREE Hours

Candidates should answer SIX parts of SECTION A, and TWO questions from SECTION B.

Separate answer books must be used for each Section of the paper.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper. Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

SECTION A – Answer any SIX parts of this section

1.1) A certain analytic function $f(z) = u(x, y) + iv(x, y)$ has a real part

$$u(x, y) = x^2 + 4x - y^2 + 2y\,.$$

Use the Cauchy-Riemann equations to determine the imaginary part $v(x, y)$ of this function.

[7 marks]

1.2) Determine all the values of the number $\mathrm{Re}\,(-1 + i)^i$.

[7 marks]

1.3) Locate and classify all the singularities in the finite z plane of the function

$$f(z) = \frac{(z^2 + z - 2)\sin z}{z^2(z-1)(z-4)^2}\,.$$

[7 marks]

1.4) Determine the Laurent series for the function

$$f(z) = \frac{1}{z(z-1)(z-2)}$$

which is valid in the region $0 < |z| < 1$.

[7 marks]

1.5) Show that the Gamma function

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt\,,$$

where $\mathrm{Re}(z) > 0$, satisfies the functional equation

$$\Gamma(z+1) = z\Gamma(z)\,.$$

[7 marks]

1.6) Use the Bessel function series

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)} \left(\frac{z}{2}\right)^{2m+\nu} ,$$

where $\Gamma(x)$ denotes the gamma function, to derive the relation

$$\frac{d}{dz}\left[z^{-\nu}\, J_\nu(z)\right] = -z^{-\nu}\, J_{\nu+1}(z)\,.$$

[7 marks]

1.7) A dynamical system with n degrees of freedom has a Hamiltonian

$$\mathcal{H}(q_1, \ldots, q_n; p_1, \ldots, p_n; t),$$

where the symbols $\{q_\alpha, p_\alpha; \alpha = 1, \ldots, n\}$ and t have their usual meaning. Show that a dynamical function $F = F(q_1, \ldots, q_n; p_1, \ldots, p_n)$ will be a constant of the motion if the Poisson bracket of F with the Hamiltonian is zero.

[7 marks]

1.8) Use the method of Lagrange multipliers to find the extremum values of the function

$$f(x, y) = xy,$$

where the variables x and y are subject to the constraint

$$y + 4x - xy = 0.$$

[7 marks]

SECTION B – Answer TWO questions in this section

2) State the *residue theorem* for evaluating contour integrals in complex analysis. Describe three methods that can be used to calculate residues.

[8 marks]

Use the residue theorem to evaluate the following definite integrals:

$$\text{(a)} \qquad \int_C \frac{z-\pi}{z^2 \sin z}\, dz \ ,$$

where the contour C is the unit circle $|z| = 1$ described in the positive sense.

[8 marks]

$$\text{(b)} \qquad \int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)^2}\, dx \ .$$

[14 marks]

In part (b), justification should be given for the neglect of any contour integral which is not taken along the real axis.

3) A circular stretched membrane of radius a lies in a region of the xy plane with plane polar coordinates $0 \le \rho \le a$ and $0 \le \phi < 2\pi$. The membrane has all its boundary edges clamped in the xy plane. When the membrane is allowed to vibrate freely with small amplitude the vertical displacement ψ of the membrane satisfies the equation

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} \ ,$$

where c is a constant. Show that the normal modes of vibration of the membrane are

$$\psi_{m,s}(\rho,\phi,t) = A_{m,s}\, J_m(k_{m,s}\, \rho)\, \sin(m\phi + B_{m,s})\, \cos(ck_{m,s}t + D_{m,s}) \ ,$$

where $m = 0, 1, 2, \ldots$, $s = 1, 2, \ldots$, $A_{m,s}$, $B_{m,s}$ and $D_{m,s}$ are constants, $k_{m,s} = j_{m,s}/a$ and $\{j_{m,s}; s = 1, 2, \ldots\}$ are the positive zeros of the Bessel function $J_m(z)$.

[20 marks]

Show that the radial part of the normal mode $\psi_{m,s}(\rho,\phi,t)$ satisfies the orthogonality relation

$$\int_0^a J_m\left(j_{m,r}\frac{\rho}{a}\right) J_m\left(j_{m,s}\frac{\rho}{a}\right) \rho\, d\rho = 0 \ ,$$

where $r, s = 1, 2, \ldots$ and $r \neq s$.

[10 marks]

[It may be assumed that $J_m(z)$ is a solution of the differential equation

$$z^2\, w'' + z\, w' + (z^2 - m^2)w = 0 \ . \ ]$$

4) Derive the Hamilton canonical equations of motion for a classical system which has a Lagrangian $L(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t)$ corresponding to n degrees of freedom.

[9 marks]

A particle of mass m is constrained to move on the surface of a smooth torus which has a parametric representation

$$x = \rho \cos\psi, \quad y = \rho \sin\psi, \quad z = b \sin\theta\,,$$

where

$$\rho = a + b\cos\theta\,, \quad (a > b > 0)$$

with $0 \leq \phi < 2\pi$, and $0 \leq \theta < 2\pi$. No external forces act on the particle. Using θ and ϕ as generalised cordinates show that the Lagrangian for the system is

$$L = \frac{m}{2}\left[(a + b\cos\theta)^2\,\dot{\psi}^2 + b^2\,\dot{\theta}^2\right]\,.$$

[8 marks]

Determine the Hamiltonian for the system and hence derive the Hamilton canonical equations of motion for the particle.

[7 marks]

Show that if the particle moves round the outer equatorial circle of the torus with $\theta = 0$, then $\dot{\psi}$ must be a constant of the motion. Discuss **briefly** what would happen if a **small** disturbance is made to this equatorial motion.

[6 marks]

5) A functional $J : A^2(x_0, x_1) \to R^1$ is defined by

$$J[y] = \int_{x_0}^{x_1} F(x, y, y')\, dx\,,$$

where the function $F(x, y, y')$ has continuous second-order derivatives with respect to all its arguments, R^1 denotes a real number and $y' = dy/dx$. The class $A^2(x_0, x_1)$ of admissible functions consists of all functions $y(x)$ which have a continuous second-order derivative for $x_0 \le x \le x_1$ and have the same fixed end-point values $y(x_0) = y_0$ and $y(x_1) = y_1$. Prove that if $y(x) \in A^2(x_0, x_1)$ gives an extremum to $J[y]$ then it must satisfy the differential equation

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0\,.$$

[12 marks]

Hence show that if $F(x, y, y')$ does not depend explicitly on the variable x, then the extremal function $y(x)$ also satisfies the equation

$$F - y'\frac{\partial F}{\partial y'} = C\,,$$

where C is a constant.

[6 marks]

Determine the extremal function $y(x)$ of the functional

$$J[y] = \int_{x_0}^{x_1} \frac{1}{y}\left[1 + (y')^2\right]^{1/2} dx$$

which passes through the end-points $(x_0, y_0) = (0, 1)$ and $(x_1, y_1) = (1, 2)$.
[12 marks]