King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP3201 Mathematical Methods in Physics III

Summer 2005

Time allowed: THREE Hours

Candidates should answer all SIX parts of SECTION A, and no more than TWO questions from SECTION B. No credit will be given for answering further questions.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper. Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

Physical Constants

Permittivity of free space	ϵ_0	=	8.854×10^{-12}	$\mathrm{F\,m^{-1}}$
Permeability of free space	μ_0	=	$4\pi \times 10^{-7}$	$\mathrm{H\,m^{-1}}$
Speed of light in free space	c	=	2.998×10^{8}	$\mathrm{m\,s^{-1}}$
Gravitational constant	G	=	6.673×10^{-11}	$\mathrm{N\,m^2\,kg^{-2}}$
Elementary charge	e	=	1.602×10^{-19}	C
Electron rest mass	m_{e}	=	9.109×10^{-31}	kg
Unified atomic mass unit	m_{u}	=	1.661×10^{-27}	kg
Proton rest mass	m_{p}	=	1.673×10^{-27}	kg
Neutron rest mass	m_{n}	=	1.675×10^{-27}	kg
Planck constant	h	=	6.626×10^{-34}	$\mathrm{J\,s}$
Boltzmann constant	k_{B}	=	1.381×10^{-23}	$\mathrm{J\,K^{-1}}$
Stefan-Boltzmann constant	σ	=	5.670×10^{-8}	$\mathrm{W\,m^{-2}\,K^{-4}}$
Gas constant	R	=	8.314	$\mathrm{J\,mol^{-1}\,K^{-1}}$
Avogadro constant	N_{A}	=	6.022×10^{23}	$\mathrm{mol^{-1}}$
Molar volume of ideal gas at STP		=	2.241×10^{-2}	$\mathrm{m^3}$
One standard atmosphere	P_0	=	1.013×10^{5}	$\mathrm{N\,m^{-2}}$

Bessel's equation:

$$x^2 \frac{d^2}{dx^2} J_p\left(x\right) + x \frac{d}{dx} J_p\left(x\right) + \left(x^2 - p^2\right) J_p\left(x\right) = 0$$

Laplacian in polar co-ordinates:

$$\nabla^2 u\left(r,\theta\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} u\left(r,\theta\right)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} u\left(r,\theta\right)$$

Euler's equation:

$$\delta \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} dy \; F\left(x, y, h\left(x,y\right), \frac{\partial h\left(x,y\right)}{\partial x}, \frac{\partial h\left(x,y\right)}{\partial y}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial x}\frac{\partial F}{\partial\left(\frac{\partial h}{\partial x}\right)} + \frac{\partial}{\partial y}\frac{\partial F}{\partial\left(\frac{\partial h}{\partial y}\right)} - \frac{\partial F}{\partial h} = 0$$

SECTION A – Answer all SIX parts of this section

1.1) Consider a potential $V(x,y)$ satisfying the Laplace equation in the plane . It is given on the x-axis by

$$V(x, y=0) = V_0 e^{-\frac{x^2}{a^2}}$$

where V_0 and a are real.

Using the theory of analytic functions show that

$$V(x,y) = V_0 e^{-\frac{x^2-y^2}{a^2}} \cos\left(\frac{2xy}{a^2}\right).$$

[7 marks]

1.2) By transforming the path of integration in the complex k-plane to the horizontal line $\mathrm{Im}(k) = -i\frac{x}{2\kappa t}$, show that the integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\kappa k^2 t - ikx} dk$$

can be rewritten as

$$\frac{e^{-\frac{x^2}{4\kappa t}}}{2\pi}\int_{-\infty}^{\infty} e^{-\kappa k^2 t} dk.$$

[7 marks]

1.3) Given that

$$u(x,y) = x^2 + 6x - y^2$$

is the real part of an analytic function $f(z)$, derive the corresponding imaginary part of $f(z)$ by use of the Cauchy-Riemann relations.

[7 marks]

1.4) On writing the Bessel function $J_\nu(x)$ (see the rubric) as $x^\alpha g_\nu(x)$ show that the differential equation for $g_\nu(x)$ does not involve first derivatives when $\alpha = -\frac{1}{2}$ and derive the resulting differential equation for $g_\nu(x)$.

When $\nu^2 = {}^1\!/_4$ what is the general solution for $J_{1/2}(x)$?

[7 marks]

1.5) Find the Laurent series about $z = 1$ for

$$\frac{e^z}{e\,(z-1)^2}.$$

[7 marks]

1.6) Evaluate

$$\oint_C \frac{dz}{(z-3)(2z-1)}$$

when C is the circle $|z| = 1$.

[7 marks]

SECTION B – Answer TWO questions

2) The surface of a circular drum of radius R is a two-dimensional membrane. Let $u(r, \varphi)$ be the displacement out of the plane of the membrane due to a wave at a point on its surface. The point (r, φ) is expressed in polar co-ordinates with respect to the centre of the drum with $0 \leq r \leq R$ and $0 \leq \varphi < 2\pi$. $u(r, \varphi)$ satisfies the Helmholtz equation

$$\nabla^2 u + k^2 u = 0$$

where k is the wavenumber of the wave. At the edge of the drum $r = R$, the membrane is stationary and so $u(R, \varphi) = 0$.

a) On writing $u(r, \varphi) = F(r)\, G(\varphi)$ show that the Helmholtz equation reduces to

$$\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} + \left(k^2 - \frac{\mu}{r^2}\right) F = 0$$

and

$$\frac{d^2G}{d\varphi^2} + \mu G = 0$$

where μ is a constant. [12 marks]

b) Justify the following boundary conditions:

$$\begin{cases} F(R) = 0 \\ G(\varphi) = G(\varphi + 2\pi) \\ F(r) < \infty. \end{cases}$$

Hence show that $\mu = m^2$ where m is an integer. [8 marks]

c) Prove that the general solution for $F(r)$ has the form

$$F(r) = AJ_m(kr) + BN_m(kr)$$

in terms of constants of integration A and B. [6 marks]

From the boundary conditions deduce any constraints on the values of A, B and k.

[4 marks]

3a) Consider a meromorphic function $h(z)$ which can be expanded around a point $z = z_0$ as a Laurent series, i.e.

$$h(z) = \sum_{n=-\infty}^{\infty} a_n \left(z - z_0\right)^n .$$

Show that

$$\oint\limits_{C_\varepsilon} h\left(z\right) dz = \sum_{n=-\infty}^{\infty} i a_n \varepsilon^{n+1} \int\limits_{0}^{2\pi} e^{i(n+1)\varphi} d\varphi$$

where C_ε is a circle centred at $z = z_0$ of small radius ε. [5 marks]

b) Prove by direct integration that

$$\int\limits_{0}^{2\pi} d\varphi \, e^{im\varphi} = \begin{cases} 0 & \text{for} \quad m \neq 0 \\ 2\pi & \text{for} \quad m = 0. \end{cases}$$

[4 marks]

c) Use this result to derive

$$\oint\limits_{C_\varepsilon} h\left(z\right) dz = 2\pi i a_{-1}.$$

. [3 marks]

d) For an arbitrary closed contour C discuss the conditions for

$$\oint\limits_{C} h\left(z\right) dz = 2\pi i a_{-1}.$$

[4 marks]

e) Prove that the function

$$h\left(z\right) = \frac{1}{1 + z^2}$$

has poles at $z = \pm i$ and that the residue at the pole $z = i$ is $\frac{1}{2i}$. [7 marks]

f) Use the Cauchy residue theorem to show that

$$\int\limits_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx = \pi.$$

[7 marks]

4a) Outline the use of analytic functions and the Cauchy-Riemann relations to solve the two-dimensional Laplace's equation. [10 marks]

b) An arbitrary point on the surface of a soap film is given by the position vector

$$\overrightarrow{r} = \begin{Bmatrix} x \\ y \\ h(x,y) \end{Bmatrix}.$$

The projection of the soap film on the $x - y$ plane is a square R :

$$R = \{-1 \leq x \leq 1,\ -1 \leq y \leq 1\}.$$

The boundary conditions satisfied by $h(x,y)$ are

$$h(x,1) = h(x,-1) = x^2 - 1$$
$$h(1,y) = h(-1,y) = 1 - y^2.$$

An infinitesimal surface area dS of the soap film is

$$dS = \left| \frac{\partial \overrightarrow{r}}{\partial x} \times \frac{\partial \overrightarrow{r}}{\partial y} \right| dxdy.$$

In terms of $h(x,y)$ show that

$$dS = \sqrt{1 + |\nabla h|^2} dxdy$$

where ∇h is the two-dimensional vector $\nabla h = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right)$. [6 marks]

c) The shape of the soap film is determined by minimising its area S

$$S = \int_{-1}^{1} dx \int_{-1}^{1} dy \sqrt{1 + |\nabla h|^2}.$$

From the Euler-Lagrange equations show that $h(x,y)$ satisfies the two-dimensional Laplace's equation

$$\nabla^2 h = 0$$

when $|\nabla h| \ll 1$. [10 marks]

Hence by choosing a suitable analytic function demonstrate that the shape of the soap film is given by

$$h(x,y) = x^2 - y^2.$$

[4 marks]