King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP3201 Mathematical Methods in Physics III

Summer 2004

Time allowed: THREE Hours

Candidates should answer no more than SIX parts of SECTION A, and no more than TWO questions from SECTION B. No credit will be given for answering further questions.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper. Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

Physical Constants

Permittivity of free space	ϵ_0	=	8.854×10^{-12}	$\mathrm{F\,m^{-1}}$
Permeability of free space	μ_0	=	$4\pi \times 10^{-7}$	$\mathrm{H\,m^{-1}}$
Speed of light in free space	c	=	2.998×10^{8}	$\mathrm{m\,s^{-1}}$
Gravitational constant	G	=	6.673×10^{-11}	$\mathrm{N\,m^2\,kg^{-2}}$
Elementary charge	e	=	1.602×10^{-19}	C
Electron rest mass	m_{e}	=	9.109×10^{-31}	kg
Unified atomic mass unit	m_{u}	=	1.661×10^{-27}	kg
Proton rest mass	m_{p}	=	1.673×10^{-27}	kg
Neutron rest mass	m_{n}	=	1.675×10^{-27}	kg
Planck constant	h	=	6.626×10^{-34}	$\mathrm{J\,s}$
Boltzmann constant	k_{B}	=	1.381×10^{-23}	$\mathrm{J\,K^{-1}}$
Stefan-Boltzmann constant	σ	=	5.670×10^{-8}	$\mathrm{W\,m^2\,K^{-4}}$
Gas constant	R	=	8.314	$\mathrm{J\,mol^{-1}\,K^{-1}}$
Avogadro constant	N_{A}	=	6.022×10^{23}	$\mathrm{mol^{-1}}$
Molar volume of ideal gas at STP		=	2.241×10^{-2}	$\mathrm{m^3}$
One standard atmosphere	P_0	=	1.013×10^{5}	$\mathrm{N\,m^{-2}}$

Relations involving Bessel function relations of the first and second kind:

$$\frac{d}{du}\left(u^{-p}J_p\left(u\right)\right) = -u^{-p}J_{p+1}\left(u\right)$$

$$\frac{d}{du}\left(u^{-p}Y_p\left(u\right)\right) = -u^{-p}Y_{p+1}\left(u\right)$$

$$J_p\left(u\right)Y_{p+1}\left(u\right) - J_{p+1}\left(u\right)Y_p\left(u\right) = -\frac{2}{\pi u}$$

Euler-Lagrange equations in terms of generalised co-ordinates q_α and Lagrangian L:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial q_\alpha} = 0.$$

SECTION A – Answer SIX parts of this section

1.1) A certain analytic function $f(z) = u(x, y) + iv(x, y)$ has a real part

$$u(x, y) = x^2 - y^2 + 4x + 3y.$$

Use the Cauchy-Riemann relations to determine the imaginary part $v(x, y)$ of this function.

[7 marks]

1.2) Determine all the values of the expression $Re[(-1+i)^{2i}]$.

[7 marks]

1.3) Locate and classify all singularities in the finite z plane of the function

$$f(z) = \frac{(z^2 + z - 2)\sin^2 z}{z^3 (z-1)(z-4)^2}.$$

[7 marks]

1.4) Determine the Laurent series for the function

$$f(z) = \frac{1}{z(z-1)(z+2)}$$

which is valid in the region $0 < |z| < 1$.

[7 marks]

1.5) Show that the Gamma function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt,$$

where $Re\ z > 0$, satisfies the functional equation

$$\Gamma(z+2) = (z+1) z \Gamma(z).$$

[7 marks]

1.6) Derive the relation

$$\frac{d}{dz}\left(z^{-\nu}J_{\nu}\left(z\right)\right)=-z^{-\nu}J_{\nu+1}\left(z\right)$$

using the Bessel function series

$$J_{\nu}\left(z\right)=\sum_{m=0}^{\infty}\frac{\left(-1\right)^{m}}{m!\Gamma\left(m+\nu+1\right)}\left(\frac{z}{2}\right)^{2m+\nu}.$$

[7 marks]

1.7) Evaluate

$$\int_{0}^{2\pi}\frac{d\theta}{5+4\cos\theta}$$

by showing that it can be written as a contour integral

$$\frac{1}{i}\oint_{C}\frac{dz}{\left(2z+1\right)\left(z+2\right)}$$

where C is the unit circle around the origin.

[7 marks]

1.8) A particle of mass m is constrained to move in the x- direction in a potential $V\left(x\right)$. Construct the Lagrangian and consequently deduce the equation of motion from the Euler-Lagrange equations (see rubric).

[7 marks]

SECTION B – Answer TWO questions

2) A mass M_2 hangs at one end of a string which passes over a fixed frictionless non-rotating pulley. The other end of this string is fixed to a frictionless non-rotating pulley of mass M_1 over which there is a string carrying masses m_1 and m_2. Let X_1 and X_2 be the distances of the masses M_1 and M_2 respectively below the centre of the fixed pulley. Let x_1 and x_2 be the distances of the masses m_1 and m_2 respectively below the centre of the movable pulley.

   Show that the kinetic and potential energies, T and V, of the system are

$$T = \frac{1}{2}M_1\dot{X}_1^2 + \frac{1}{2}M_2\dot{X}_2^2 + \frac{1}{2}m_1\left(\dot{X}_1 + \dot{x}_1\right)^2 + \frac{1}{2}m_2\left(\dot{X}_1 + \dot{x}_2\right)^2$$

   and

$$V = -M_1 g X_1 - M_2 g X_2 - m_1 g\left(X_1 + x_1\right) - m_2 g\left(X_1 + x_2\right).$$

[11 marks]

   Construct the Lagrangian L of the system including the constraints that $X_1 + X_2$ and $x_1 + x_2$ are both constant, which reflects the fact that the strings are of fixed and unequal length.

[10 marks]

   By using the Euler-Lagrange equations (see rubric) show that the acceleration of mass M_1 is

$$\frac{\left(M_1 - M_2\right)\left(m_1 + m_2\right) + 4m_1m_2}{\left(M_1 + M_2\right)\left(m_1 + m_2\right) + 4m_1m_2}g.$$

[9 marks]

3) A slab of material of constant thermal conductivity is located in the region

$$-\tfrac{1}{2} \le x \le \tfrac{1}{2}$$
$$0 \le y < \infty.$$

The temperature T at the boundaries of the slab are given by

$$T = \begin{cases} 1, & x = -\tfrac{1}{2} \\ 2, & x = \tfrac{1}{2} \\ 0, & y = 0 \end{cases}$$

Show that the mapping

$$w = \sin(\pi z)$$

transforms the region of the slab (when the $x - y$ plane is considered as the complex plane) into the upper half of the $w-$plane.

[15 marks]

By solving the Laplace equation in the $w-$plane and on using the mapping, prove that in the steady state

$$T = \frac{1}{\pi}\tan^{-1}\left\{\frac{\cos\pi x\ \sinh\pi y}{\sin\pi x\ \cosh\pi y\ +1}\right\} - \frac{2}{\pi}\tan^{-1}\left\{\frac{\cos\pi x\ \sinh\pi y}{\sin\pi x\ \cosh\pi y\ -1}\right\} + 2.$$

[15 marks]

4) State the residue theorem for evaluating contour integrals in complex analysis. Describe one general method that can be used to calculate residues.

[8 marks]

Use the residue theorem to evaluate the following definite integrals:

(*a*)

$$\oint_C \frac{z-\pi}{z\sin z}\,dz,$$

where the contour C is the unit circle $|z| = 1$ described in the positive sense.

[8 marks]

(*b*)

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)\,(x^2+9)^2}\,dx.$$

Justification should be given for the neglect of any contour integral which is not taken along the real axis.

[14 marks]

5) A simple pendulum of mass m hangs on a string of length l which increases at a steady rate, that is $l = l_0 + vt$. Let θ be the angular deviation of the pendulum from the vertical.

Show that the kinetic energy of the pendulum is

$$\frac{1}{2}m\left(\dot{l}^2 + l^2\dot{\theta}^2\right).$$

[3 marks]

Write the Lagrangian for the pendulum and from the Euler-Lagrange equations show that

$$\frac{d}{dt}\left(ml^2\dot{\theta}\right) + mgl\sin\theta = 0$$

where g is the acceleration of gravity.

[4 marks]

For small θ (where $\sin\theta$ is well approximated by θ) deduce that

$$l\frac{d^2\theta}{dl^2} + 2\frac{d\theta}{dl} + \frac{g}{v^2}\theta = 0.$$

[3 marks]

Show that the solution of this equation which satisfies the boundary conditions that $\theta = \theta_0$ and $\frac{d\theta}{dt} = 0$ when $t = 0$ is

$$\theta = \frac{\pi u_0^2\theta_0}{2u}\left[-Y_2(u_0)J_1(u) + J_2(u_0)Y_1(u)\right].$$

where $u = \frac{2(gl)^{1/2}}{v}$ and $u_0 = \frac{2(gl_0)^{1/2}}{v}$.

Hint:

$$\frac{d^2y}{dx^2} + \frac{1-2a}{x}\frac{dy}{dx} + \left[\left(bcx^{c-1}\right)^2 + \frac{a^2 - p^2c^2}{x^2}\right]y = 0$$

has the solution

$$y = x^a\left(AJ_p(bx^c) + BY_p(bx^c)\right)$$

where J_p and Y_p are Bessel functions of the first and second kind and A and B are constants. See also the rubric for other Bessel function relations.

[20 marks]