Solutions to 2005 MathII Exam Paper

March 30, 2006

Solutions

$\left(\frac{1}{\sqrt{2}}\right)^2 (\hat{\imath}+\hat{\jmath})(\hat{\imath}+\hat{\jmath}) = \frac{1}{2}(\hat{\imath}\hat{\imath}+\hat{\jmath}\hat{\jmath}) = 1$

1.1 From $\alpha = x+y$, $\beta = x-y$, $\gamma = z$ we obtain

$$x = \frac{1}{2}(\alpha+\beta), \quad y = \frac{1}{2}(\alpha-\beta), \quad z = \gamma$$

Therefore, $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$\frac{\partial \vec{r}}{\partial \alpha} = \frac{1}{2}(\hat{\imath}+\hat{\jmath}); \; h_\alpha = \left|\frac{\partial \vec{r}}{\partial \alpha}\right| = \frac{1}{\sqrt{2}} \rightarrow \boxed{\hat{e}_\alpha = \frac{1}{\sqrt{2}}(\hat{\imath}+\hat{\jmath})}$$

$$\frac{\partial \vec{r}}{\partial \beta} = \frac{1}{2}(\hat{\imath}-\hat{\jmath}); \; h_\beta = \left|\frac{\partial \vec{r}}{\partial \beta}\right| = \frac{1}{\sqrt{2}} \rightarrow \boxed{\hat{e}_\beta = \frac{1}{\sqrt{2}}(\hat{\imath}-\hat{\jmath})}$$

$$\boxed{\hat{e}_\gamma = \hat{k}}$$

1

4

Orthogonality: $\hat{e}_\alpha \cdot \hat{e}_\beta = \frac{1}{2}(1-1) = 0$

$$e_\alpha \cdot e_\gamma = 0$$

$$e_\beta \cdot e_\gamma = 0$$

2

yes! (in fact, the new system represents a rotated Cartesian system → not req. for a student though!)

1.2 Using ~~change the order~~ rewrite the double integral:

$$\text{2nd} = \int_{-\infty}^{\infty} dx \; H(x-1)\, e^{-x^2} \underbrace{\left[\int_{-\infty}^{\infty} dy \, e^{2\pi i x y}\right]}_{\delta(x)} =$$

1

$$= \int_{-\infty}^{\infty} dx \; \delta(x)\, e^{-x^2} H(x-1)$$

2

$$= \left[e^{-x^2} H(x-1)\right]_{x=0} = \left[H(x-1)\right]_{x=0}.$$

Since $H(x-1) = 0$ for $x=0$, we obtain zero.

1

The single integral:

$$1st = \int_{-\infty}^{\infty} dx \, \underbrace{H(x-1)} e^{-x} = \int_{1}^{\infty} e^{-x} \, dx =$$ (2)

$$= + e^{-x} \Big|_{\infty}^{1} = +e^{-1} = \frac{1}{e}$$ (1)

Answer: $1/e$

$\boxed{1.3}$ $f(t) = t e^{-\alpha|t|} = \begin{cases} t e^{-\alpha t}, & t \geq 0 \\ t e^{\alpha t}, & t < 0 \end{cases}$ (1)

$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t) \, e^{i2\pi\nu t} \, dt$$ (1)

$$= \int_{-\infty}^{\infty} t e^{-\alpha|t|} e^{i2\pi\nu t} dt = \underbrace{\int_{-\infty}^{0} t e^{\alpha t} e^{i2\pi\nu t} \, dt}_{I_1}$$

$$+ \underbrace{\int_{0}^{\infty} t e^{-\alpha t} e^{i2\pi\nu t} dt}_{I_2} =$$ (1)

$$I_2 = \int_0^{\infty} \underbrace{t}_{u} \underbrace{e^{-(\alpha - i2\pi\nu)t} dt}_{dv} = \left| \text{by parts} \right| =$$

$$= \underbrace{t \frac{1}{-(\alpha - i2\pi\nu)} e^{-(\alpha - i2\pi\nu)t} \Big|_0^{\infty}}_{=0} + \frac{1}{\alpha - i2\pi\nu} \int_0^{\infty} e^{-(\alpha - i2\pi\nu)t} dt$$

$$= \frac{1}{\alpha - i2\pi\nu} \cdot \frac{1}{-(\alpha - i2\pi\nu)} e^{-(\alpha - i2\pi\nu)t} \Big|_0^{\infty}$$ (2)

$$= \frac{1}{(\alpha - i2\pi\nu)^2}$$

$$I_1 = \left| \text{change variables } t \to -t \right| = \int_{\infty}^{0} (-t) e^{-\alpha t} e^{-i2\pi\nu t} (-dt) =$$

$$= -\int_0^\infty t e^{-\alpha t} e^{-i2\pi\nu t} = -I_1^* = -\frac{1}{(\alpha+i2\pi\nu)^2}$$ 1

Thus,

$$\mathcal{F}[f(t)] = \frac{1}{(\alpha-i2\pi\nu)^2} - \frac{1}{(\alpha+i2\pi\nu)^2} = \frac{(\alpha+i2\pi\nu)^2-(\alpha-i2\pi\nu)^2}{(\alpha^2+4\pi^2\nu^2)^2}$$

$$= \frac{8\pi\nu\alpha i}{(\alpha^2+4\pi^2\nu^2)^2}$$ 1

$\boxed{1.4}$ $$p(x) = \frac{x+2}{x^2-4} = \frac{\cancel{(x+2)}}{(x-2)\cancel{(x+2)}} = \frac{1}{x-2}$$ 1

s.p. $x = 2$

$$q(x) = \frac{x-2}{x^2-4} = \frac{\cancel{x-2}}{(x+2)\cancel{(x-2)}} = \frac{1}{x+2}$$ 1

s.p. $x = -2$

Thus, there are two singular points $x = \pm 2$. 1

Classification:

(a) ~~$(x-2)p(x) =$~~ $\underline{x=2}$

$$(x-2)p(x) = \frac{x-2}{x-2} = 1 \text{ , ok at } x=2$$

$$(x-2)^2 q(x) = \frac{(x-2)^2}{(x+2)} = 0 \text{ , ok.}$$ 2

→ regular s.p.

(b) $\underline{x=-2}$

$$(x+2)\,p(x) = \frac{x+2}{x-2} = 0, \text{ ok.}$$

$$(x+2)^2\,q(x) = \frac{(x+2)^2}{x+2} = x+2 = 0, \text{ ok.}$$

$\rightarrow$ RSP

2

Thus, both points are regular singular points.

$\boxed{1.5}$ • Using the given expression for Δ in the spherical coordinates, and ignoring ~~the~~ dependence on angles θ and ϕ, we obtain:

$$\frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) = \cancel{\ldots} = \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial r^2}$$

2

• Using method of separation of variables:

$$\psi(r,t) = R(r)\,T(t)$$

1

$$T\,\frac{1}{r^2}\frac{d}{dr}\left(r^2R'\right) = R\,\frac{1}{v^2}\frac{d^2T}{dt^2}$$

Divide by TR:

$$\underbrace{\frac{1}{Rr^2}\left(r^2R'\right)'}_{\text{depends only on } r} = \underbrace{\frac{1}{v^2}\frac{T''}{T}}_{\text{depends only on } t}$$

or $+k$, also accepted!

2

$$\hookrightarrow \frac{1}{Rr^2}\left(r^2R'\right)' = -k \;\rightarrow\; \left(r^2R'\right)' + kr^2R = 0$$

1

$$\boxed{R'' + \frac{2}{r}R' + kR = 0} \xleftarrow{\text{or}} \boxed{r^2R'' + 2rR' + kr^2R = 0} \quad \text{ODE for } R(r)$$

$$\hookrightarrow \frac{1}{v^2}\frac{T''}{T} = -k \;\rightarrow\; \boxed{T'' + kv^2T = 0} \quad \text{ODE for } T(t)$$

1

k – separation constant

1.6 Def.

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt, \quad \mathrm{Re}\, s > \sigma_0 \qquad 2$$

Calculation:

$$\sinh(\alpha t) = \frac{1}{2}\left(e^{\alpha t} - e^{-\alpha t}\right)$$

$$\mathcal{L}[\sinh(\alpha t)] = \frac{1}{2}\mathcal{L}[e^{\alpha t}] - \frac{1}{2}\mathcal{L}[e^{-\alpha t}] \qquad 1$$

$$\mathcal{L}[e^{\alpha t}] = \int_0^\infty e^{-(s-\alpha)t}\,dt = \frac{1}{s-\alpha}\, e^{-(s-\alpha)t}\Big|_\infty^0 = \frac{1}{s-\alpha}$$

$$(\mathrm{Re}(s-\alpha) > 0) \qquad 2$$

Hence

$$\mathcal{L}[e^{-\alpha t}] = \frac{1}{s+\alpha} \qquad 1$$

and

$$\mathcal{L}[\sinh(\alpha t)] = \frac{1}{2}\frac{1}{s-\alpha} - \frac{1}{2}\frac{1}{s+\alpha} = \frac{s+\alpha-s+\alpha}{2(s^2-\alpha^2)} = \frac{2\alpha}{2(s^2-\alpha^2)}$$

$$= \frac{\alpha}{s^2-\alpha^2} \qquad 1$$

$\boxed{2}$ (a) $4x^2y'' - 2x(x+2)y' + (x+3)y = 0$

$$p(x) = \frac{-2x(x+2)}{4x^2} = -\frac{x+2}{2x}$$

$$q(x) = \frac{x+3}{4x^2}$$

$x=0$ - singular point.

2

Classification:

$$xp = -\frac{x+2}{2}, \text{ ok.}$$

$$x^2q = \frac{x+3}{4}, \text{ ok.}$$

$\rightarrow$ RSP

(B) Consequently, can use the method of Frobenius and expand about the $x=0$ point:

$$y = \sum_{n=0}^{\infty} a_n x^{n+s}, \quad s - \text{unknown parameter}$$

We have:

$$y' = \sum_{n=0}^{\infty} a_n (n+s) x^{n+s-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+s)(n+s-1) x^{n+s-2}$$

1

Substitute into the DE:

$$\sum_{n=0}^{\infty} a_n 4(n+s)(n+s-1)x^{n+s} - \sum_{n=0}^{\infty} a_n 2(n+s)(x+2)x^{n+s} + \sum_{n=0}^{\infty} a_n (x+3)x^{n+s} = 0$$

1

Rearrange:

$$\sum_{n=0}^{\infty} a_n\left[4(n+s)(n+s-1) - 4(n+s) + 3\right]x^{n+s} - \sum_{n=0}^{\infty} a_n\left[2(n+s) - 1\right]x^{n+s+1} = 0$$

The 1st sum starts from x^s, the 2nd - from x^{s+1}, so that we can separate out the 1st term in the 1st sum and also change the summation index in the 1st sum:

$$a_0\left[4s(s-1)-4s+3\right]x^s+\sum_{n=1}^{\infty}a_n\left[4(n+s)(n+s-2)+3\right]x^{n+s}$$

$$-\sum_{n=0}^{\infty}a_n\left[2n+2s-1\right]x^{n+s+1}=0$$

In the 1st sum: $n \mapsto n^* = n'+1$

$n = 1, 2, \ldots$

$n' = 0, 1, \ldots$ as in the 2nd sum

We obtain:

$$a_0\left[4s^2-8s+3\right]+\sum_{n'=0}^{\infty}a_{n'+1}\left[4(n'+1+s)(n'+s-1)+3\right]x^{n'+s+1}$$

$$-\sum_{n=0}^{\infty}a_n(2n+2s-1)x^{n+s+1}=0$$

Combine the both sums $(n' \to n)$:

$$a_0(4s^2-8s+3)+\sum_{n=0}^{\infty}\Big\{a_{n+1}\left[4(n+s+1)(n+s-1)+3\right]-$$ 6

$$-\,a_n(2n+2s-1)\Big\}x^{n+s+1}=0 \qquad (*)$$

This eq. is satisfied for all x (within the convergence region/radius) if, and only if $(a_0 \neq 0)$: 1

$4s^2-8s+3=0$ ← the indicial equation 1

and, simultaneously,

$$a_{n+1}=a_n\,\frac{2n+2s-1}{4\left[(n+s)^2-1\right]+3}=a_n\,\frac{1}{2(n+s)+1}$$ 2

$n = 0, 1, 2, \ldots$

the recurrence ~~equation~~ relation.

Solving the indicial eq: $s_\pm=\dfrac{8\pm\sqrt{64-4\cdot3\cdot4}}{2\cdot4}=\dfrac{8\pm\sqrt{64-48}}{8}$

$$=\frac{8\pm\sqrt{16}}{8}=\frac{8\pm4}{8}=\begin{cases}12/8=3/2 & (\text{if } +)\\ 4/8=1/2 & (\text{if } -)\end{cases}$$

Thus, $s_1 = \frac{1}{2}$, $s_2 = \frac{3}{2}$ 1

$\boxed{s_1 = \frac{1}{2}}$ Choose $a_0 = 1$

$$a_{n+1} = a_n \frac{1}{2(n+1)}$$ 1

$n=0 \to a_1 = a_0 \frac{1}{2} = \frac{1}{2}$ 1

$n=1 \to a_2 = a_1 \frac{1}{2\cdot 2} = \frac{1}{2\cdot 2^2} = \frac{1}{(1\cdot 2)\,2^2} = \frac{1}{8}$ 1

$n=2 \to a_3 = a_2 \frac{1}{2\cdot 3} = \frac{1}{(1\cdot 2\cdot 3)\,2^3} = \frac{1}{48}$ 1

etc.

Thus, $y_1(x) = x^{1/2}\left[1 + \frac{x}{2} + \frac{x^2}{2\cdot 2^2} + \frac{x^3}{3!\,2^3} + \dots\right]$ 1

$\boxed{s_2 = \frac{3}{2}}$ The recurrence relation:

$$a_{n+1} = a_n \frac{1}{2(n+2)}$$ 1

$a_1 = a_0 \frac{1}{2\cdot 2} = \frac{1}{2\cdot 2} = \frac{1}{4}$ 1

$a_2 = a_1 \frac{1}{2\cdot 3} = \frac{1}{(1\cdot 2\cdot 3)\,2^2} = \frac{1}{3!\,2^2} = \frac{1}{24}$ 1

$a_3 = a_2 \frac{1}{2\cdot 4} = \frac{1}{(1\cdot 2\cdot 3\cdot 4)\,2^3} = \frac{1}{4!\,2^3}$, etc. $= \frac{1}{192}$ 1

$$y_2(x) = x^{3/2}\left[1 + \frac{x}{2\cdot 2} + \frac{x^2}{3!\,2^2} + \frac{x^3}{4!\,2^3} + \dots\right]$$ 1

(d) It can be seen that

$$y_1(x) = x^{1/2} + \frac{1}{2}x^{3/2}\left[1 + \frac{x}{2\cdot 2} + \frac{x^2}{3!\,2^2} + \dots\right] \equiv$$

$$\equiv x^{1/2} + \frac{1}{2}\, y_2(x),$$ 3

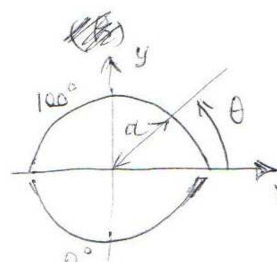
so that one can take $y_1 = x^{1/2}$ without any error.

(e) General solution:

$$y(x) = C_1\, y_1(x) + C_2\, y_2(x)$$ 2

3 For this problem it is convenient to employ
(a) polar coordinates (i.e. cylindrical coordinates without z).
On the other hand, we are looking for a steady-state solution, so that $\partial T / \partial t = 0$ as well.
Writing ΔT in cylindrical coordinates and ignoring z, we obtain the required equation for $T(r, \theta)$. (4)

(b)



$$T(a,\theta) = \begin{cases} 100^\circ, & 0 \le \theta \le \pi \\ 0^\circ, & \pi < \theta < 2\pi \end{cases}$$ (1)

and

$T(0,\theta)$ – finite for any θ ($\underbrace{\text{i.e. for } r=0}_{\text{centre}}$) (1)

(c) $T(r,\theta) = R(r)\,\Theta(\theta)$

$$\Theta\left[\frac{1}{r}R' + R''\right] + \frac{R}{r^2}\Theta'' = 0$$

Divide by ΘR and rearrange: (2)

$$\underbrace{\frac{1}{R}\left[rR' + r^2R''\right]}_{\text{depends only on } r} = \underbrace{-\frac{\Theta''}{\Theta}}_{\text{on } \theta} \rightarrow \; = +k$$

separation constant

which result in two ODE's required:

$\Theta'' + k\Theta = 0$ and $r^2R'' + rR' - kR = 0$. (2)

d) To ensure periodicity of the function $\Theta(\theta)$, k should be positive $\rightarrow$ $\Theta(\theta) = C\cos(\sqrt{k}\theta + \phi)$; C, ϕ – constants.
Also, the periodicity of 2π, in particular, is achieved if k is an integer including zero, i.e. $\sqrt{k} = n$, $n = 0, 1, 2, \ldots$
Therefore, $\Theta(\theta) = C\cos(n\theta + \phi)$ or $= A\cos n\theta + B\sin n\theta$. 4

e) Using $R(r) = r^\alpha$ in $r^2R'' + rR' - kR = 0$ gives:

$$\alpha(\alpha-1)r^\alpha + \alpha r^\alpha - n^2 r^\alpha = 0 \rightarrow \alpha^2 = n^2, \quad \alpha = \pm n.$$

Therefore, $R_1(r) = r^n$, $R_2(r) = r^{-n} = 1/r^n$. Only the 1st one should be used to ensure a finite $T(r,\theta)$ at $r = 0$. 3
Thus, $R(r) = A' r^n$ up to a factor A'.

e) The problem is linear, so that a sum of elementary solutions is the general solution:

$$T(r,\theta) = \sum_{n=0}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) r^n$$ 1

with A' being absorbed in A_n and B_n.

f) To find the constants A_n, B_n, we write $T(r,\theta)$ at $r = a$:

$$T(a,\theta) = \sum_{n=0}^{\infty} (A_n \cos n\theta + B_n \sin n\theta)\, a^n$$ 1

Use (given):

$$\int_0^{2\pi} \sin n\theta \sin m\theta \, d\theta = \int_0^{2\pi} \cos n\theta \cos m\theta \, d\theta = \frac{1}{2} 2\pi \delta_{nm} = \pi \delta_{nm}$$

$$\int_0^{2\pi} \sin n\theta \cos m\theta \, d\theta = 0$$

we obtain for $m \neq 0$: employing the boundary conditions:

$$\int_0^{2\pi} T(a,\theta) \sin m\theta \, d\theta = \sum_{n=0}^{\infty} B_n a^n (\pi \delta_{nm}) = \pi a^m B_m$$ 3

$$B_m = \frac{1}{\pi a^m} \int_0^{2\pi} T(a,\theta) \sin m\theta \, d\theta =$$

$$= \frac{100}{\pi a^m} \int_0^{\pi} \sin m\theta \, d\theta = \frac{100}{\pi a^m} \frac{1}{m} \cos m\theta \Big|_{\pi}^{0} =$$

$$= \frac{100}{\pi m a^m}\left[1-(-1)^m\right] = \frac{200}{\pi m a^m} \times \begin{cases} 2, \text{ if } m = \text{odd} \\ 0, \text{ if } m = \text{even} \end{cases} =$$

$$= \frac{200}{\pi m a^m} \begin{cases} 1, & m = \text{odd} \\ 0, & m = \text{even} \end{cases}$$

2

~~If $m \neq 0$, then~~ Similarly (still $m \neq 0$):

$$\int_0^{2\pi} T(a,\theta)\cos m\theta \, d\theta = \sum_{n=0}^{\infty} A_n a^n \pi \delta_{nm} = \pi A_m a^m,$$

2

$$A_m = \frac{1}{\pi a^m}\int_0^{2\pi} T(a,\theta)\cos m\theta\, d\theta = \frac{100}{\pi a^m}\int_0^{\pi} \cos m\theta \, d\theta =$$

2

$$= \frac{100}{\pi a^m}\,\frac{1}{m}\sin m\theta \Big|_0^{\pi} = 0.$$

2

The case $m = 0$ should be considered separately:

$$\int_0^{2\pi} T(a,\theta)\, d\theta = \sum_{n=1}^{\infty}\Big[A_n a^n \underbrace{\int_0^{2\pi} \cos n\theta\, d\theta}_{0} + B_n a^n \underbrace{\int_0^{2\pi} \sin n\theta \, d\theta}_{0}\Big] + A_0 \cdot 2\pi$$

and thus

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} T(a,\theta)\, d\theta = \frac{100}{2\pi}\int_0^{\pi} d\theta = 50$$

1

The coefficient B_0 does not appear at all as $\sin n\theta = 0$ for $n = 0$.

Finally,

$$\boxed{T(r,\theta) = 50 + \sum_{\substack{n=1 \\ (\text{odd})}}^{\infty} \frac{200}{\pi n}\left(\frac{r}{a}\right)^n \sin n\theta}$$

1

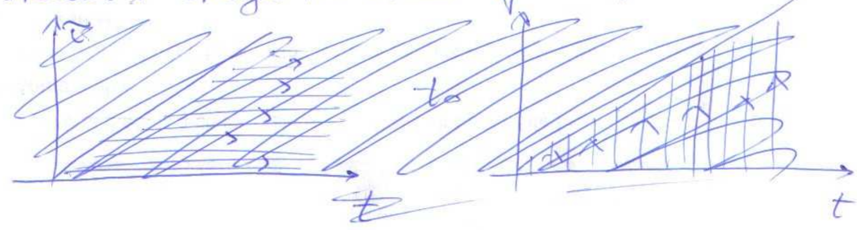
$\boxed{4}$

(a) $\mathcal{L}[e^{i\alpha t}] = \int_0^\infty e^{-(s-i\alpha)t}\,dt = \frac{1}{s-i\alpha} = \frac{s+i\alpha}{s^2+\alpha^2}$ 2

$\hookrightarrow$ $\mathcal{L}[\cos\alpha t] = \frac{s}{s^2+\alpha^2}$ and $\mathcal{L}[\sin\alpha t] = \frac{\alpha}{s^2+\alpha^2}$

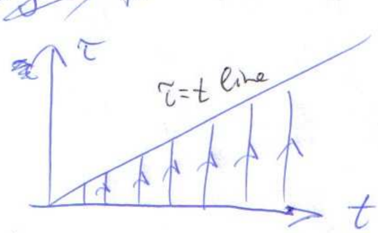
(b) $\mathcal{L}\left[\int_0^t f(t-\tau)\,g(\tau)\,d\tau\right] = \int_0^\infty dt \int_0^t d\tau\, e^{-st} f(t-\tau)\,g(\tau)$ 1

We should ~~b~~ change the order of integrals:

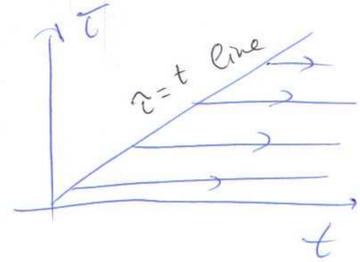


~~$\int_0^\infty d\tau \int dt$ fro~~

from



to



resulting in

$$\int_0^\infty d\tau \int_\tau^\infty dt\; e^{-st} g(\tau)\, f(t-\tau) = \left| \begin{array}{c} x = t-\tau \\ dx = dt \\ \text{in the 2nd} \\ \text{integral} \end{array} \right|$$ 4

$$= \int_0^\infty d\tau \int_0^\infty dx\; e^{-s(x+\tau)}\, g(\tau)\, f(x) =$$ 3

$$= \int_0^\infty d\tau\, e^{-s\tau} g(\tau) \cdot \int_0^\infty dx\, e^{-sx} f(x) = \mathcal{L}[g]\,\mathcal{L}[f]$$

Q.E.D.

(c) We note that

$$\frac{\alpha^2}{s(\alpha^2+s^2)} = \underbrace{\frac{1}{s}}_{\mathcal{L}[g(t)]} \cdot \underbrace{\frac{\alpha^2}{\alpha^2+s^2}}_{\mathcal{L}[f(t)]}$$ 2

But $f(t) = \alpha \sin \alpha t$ (from (a))

and $\mathcal{L}[1] = \cancel{\int_0^\infty e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^\infty} = \frac{1}{s}$

and therefore $f(t) = 1_t$. Hence:

$$\mathcal{L}^{-1}\left[\frac{\alpha^2}{s(\alpha^2+s^2)}\right] = \int_0^t 1 \cdot \alpha \cdot \sin \alpha t\, dt = \alpha \cdot \frac{1}{\alpha} \cos \alpha t \Big|_t^0 =$$ 3

$= 1 - \cos \alpha t.$ Q.E.D.

To check, directly, consider

$$\mathcal{L}[1-\cos\alpha t] = \cancel{\int_0^\infty e^{-st}\,dt} = \mathcal{L}[1] - \mathcal{L}[\cos \alpha t]$$ 2

$$= \frac{1}{s} - \frac{s}{s^2+\alpha^2} = \frac{s^2+\alpha^2-s^2}{s(s^2+\alpha^2)} = \frac{\alpha^2}{s(s^2+\alpha^2)}$$ Q.E.D.

(d) $$\mathcal{L}[f'(t)] = \int_0^\infty \underbrace{e^{-st}}_{u} \underbrace{f'(t)\,dt}_{dv} = e^{-st} f(t)\Big|_0^\infty - \int_0^\infty (-s) e^{-st} f(t)\,dt$$

$$= \underbrace{-f(0)}_{\text{since } s>0} + s\int_0^\infty e^{-st} f(t)\,dt = -f(0) + s\,\mathcal{L}[f(t)]$$ 3 Q.E.D.

(e) Introduce $Z(s) = \mathcal{L}[z(t)]$ and $Y(s) = \mathcal{L}[y(t)]$ 1
and apply the LT to both sides of the two equations:

$$\begin{cases} -z(0) + s\,Z(s) + 2Y(s) = 0 \\ -y(0) + s\,Y(s) - 2Z(s) = \frac{2}{s} \end{cases}$$ 3

~~[a student may apply the initial conditions here]~~

It is convenient to apply the initial conditions for now:

$$\begin{cases} s\,z(s) + 2y(s) = 0 \\ s\,y(s) - 2z(s) = \dfrac{2}{s} \end{cases} \rightarrow \quad y(s) = -\frac{s}{2}\,z(s)$$

$$\downarrow$$

$$\left(-\frac{s^2}{2} - 2\right) z(s) = \frac{2}{s}$$

3

and hence

$$z(s) = -\frac{2}{s}\,\frac{1}{2 + \frac{s^2}{2}} = -\,\frac{4}{s(4+s^2)}$$

Using the result from (c):

$$z(t) = -(1 - \cos 2t) = \cancel{1+}\cos 2t - 1$$ 1

Consequently,

$$y(s) = -\frac{s}{2}\,z(s) = +\,\frac{2s}{s(4+s^2)} = \frac{2}{4+s^2}$$ 1

$$y(t) = \sin 2t$$ 1