Solutions to 2003 MathII Resit Exam Paper

March 30, 2006

Solutions CP2260 August 2003

(1.1) Write,

$$\text{grad}\,\psi = \sum_{i=1}^{3} (\text{grad}\,\psi)_i \, \underline{e}_i \,, \qquad (1)$$

2

where

$$(\text{grad}\,\psi)_i = (\text{grad}\,\psi)\cdot \underline{e}_i = \frac{d\psi}{ds_i} \,, \qquad (2)$$

and $\frac{d\psi}{ds_i}$ is the rate of change of ψ along the ith coordinate line at point $P(q_1, q_2, q_3)$. For a small change $q_i \to q_i + dq_i$ along the coordinate line we have

2

$$d\psi = \left(\frac{\partial \psi}{\partial q_i}\right) dq_i \,, \qquad ds_i = h_i \, dq_i \,.$$

Hence,

2

$$\frac{d\psi}{ds_i} = \frac{1}{h_i}\left(\frac{\partial \psi}{\partial q_i}\right) , \qquad (i = 1, 2, 3) \qquad (3)$$

From (1), (2) and (3) we see that

1

$$\text{grad}\,\psi = \sum_{i=1}^{3} \frac{1}{h_i}\left(\frac{\partial \psi}{\partial q_i}\right) \underline{e}_i \,.$$

(No explanation 5)

(1.2)

$$x = r\sin\theta\cos\phi \,, \quad y = r\sin\theta\sin\phi \,, \quad z = r\cos\theta$$

Using general formulae from question 1.1:

(a)

3

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi \,, \quad \frac{\partial y}{\partial r} = \sin\theta\sin\phi \,, \quad \frac{\partial z}{\partial r} = \cos\theta$$

$$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi \,, \quad \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi \,, \quad \frac{\partial z}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi \,, \quad \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi \,, \quad \frac{\partial z}{\partial \phi} = 0$$

(B) Therefore,

2
$$\left\{\begin{aligned} h_r &= \left|\frac{\partial \vec{r}}{\partial r}\right| = \sqrt{(\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + (\cos\theta)^2} = 1 \\ h_\theta &= \sqrt{(r\cos\theta\cos\phi)^2 + (r\cos\theta\sin\phi)^2 + (-r\sin\theta)^2} = r \\ h_\phi &= \sqrt{(-r\sin\theta\sin\phi)^2 + (r\sin\theta\cos\phi)^2 + 0^2} = r\sin\theta \end{aligned}\right.$$

(C) and thus

2
$$\left\{\begin{aligned} \bar{e}_r &= \frac{1}{h_r}\left[\frac{\partial x}{\partial r}\hat{i} + \frac{\partial y}{\partial r}\hat{j} + \frac{\partial z}{\partial r}\hat{k}\right] = \sin\theta(\cos\phi\,\hat{i} + \sin\phi\,\hat{j}) + \\ &\quad + \cos\theta\,\hat{k}\ ; \\ \bar{e}_\theta &= \cos\theta(\cos\phi\,\hat{i} + \sin\phi\,\hat{j}) - \sin\theta\,\hat{k} \\ \bar{e}_\phi &= -\sin\phi\,\hat{i} + \cos\phi\,\hat{j} \end{aligned}\right.$$

(1.3) The general filtering theorem for the Dirac delta function is

2
$$\int_{-\infty}^{\infty} \delta(t-a)\, f(t)\, dt = f(a). \qquad (5)$$

To evaluate the integral

$$I = \int_{-\infty}^{\infty} \delta(4t+\pi)\,\sin(2t)\, dt\ ,$$

Let $u = 4t+\pi$, with $du = 4\,dt$. Hence, we obtain

5
$$I = \frac{1}{4}\int_{-\infty}^{\infty} \delta(u)\,.\,\sin\left(\frac{u-\pi}{2}\right) du\ . \qquad (6)$$

From (5) with $a=0$, and (6) we see that

$$I = \frac{1}{4}\sin\left(-\frac{\pi}{2}\right) = -\frac{1}{4}\ .$$

(−2 for not changing $du \leftrightarrow 4\,dt$), (just $t = -\frac{\pi}{4}$ No u trans. 1) ($t = \frac{1}{4}u$, No $dt \Rightarrow du$ included) ?

(1.4) ~~(1.4)~~ The Fourier transform is defined as

2

$$F(\nu) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-i2\pi\nu t}\, dt \,. \tag{7}$$

In the problem we have

$$F(\nu) = \int_{-\infty}^{\infty} H(t)\, e^{-4\pi t} \cdot e^{-i2\pi\nu t}\, dt \tag{8}$$

$$= \int_{0}^{\infty} 1\, e^{-(2+i\nu)2\pi t}\, dt \tag{9}$$

(2 for this stage)

(1.4) (cont.) Hence we obtain

$$F(\nu) = \left[\frac{e^{-2\pi(2+i\nu)t}}{-2\pi(2+i\nu)} \right]_0^{\infty}$$

(just missing out $e^{-4\pi t}$ 3)

5

$$= \frac{1}{2\pi(2+i\nu)} \Bigg] .$$

(1.5) $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = D\frac{\partial\phi}{\partial t}$$

$\phi = \phi(r,\theta,t) \Rightarrow$

We try the following representation for

1

$$\phi = R(r)\,\Theta(\theta)\,T(t) \tag{10}$$

which gives:

$$\frac{1}{r}\,\Theta\cdot T\cdot\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{r^2}R\cdot T\,\frac{d^2\Theta}{d\theta^2} = D\cdot R\cdot\Theta\cdot\frac{dT}{dt}$$

Divide both sides by $\Theta T R$:

1

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{r^2}\cdot\frac{1}{\Theta}\cdot\frac{d^2\Theta}{d\theta^2} = D\,\frac{1}{T}\frac{dT}{dt}$$

The LHS depends only on r, θ; the RHS – on t, thus; introducing the separation constant k_1:

1) $$\begin{cases} \dfrac{1}{rR}\dfrac{d}{dr}\left(r\dfrac{dR}{dr}\right) + \dfrac{1}{r^2}\dfrac{1}{\Theta}\dfrac{d^2\Theta}{d\theta^2} = k_1 & (11) \\ D\dfrac{1}{T}\dfrac{dT}{dt} = k_1 & (12) \end{cases}$$

which gives an ODE for $T(t)$ as:

1) $$\frac{dT}{dt} - \frac{k_1}{D}T = 0 \qquad (13)$$

Rearrange (11) as follows:

1) $$\underbrace{\frac{\cancel{r}^2}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - k_1 r^2}_{\text{depends only on } r} = \underbrace{-\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}}_{\text{depends only on } \theta} \equiv k_2$$

which gives other two equations:

1) $$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = -k_2 \rightarrow \frac{d^2\Theta}{d\theta^2} + k_2\Theta = 0 \qquad (14)$$

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - k_1 r^2 = k_2 \rightarrow$$

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) - k_1 r^2 R - k_2 R = 0$$

1) $$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) - (k_1 r^2 + k_2)R = 0 \qquad (15)$$

1.6 1.6) Consider a general DE:

$$y''(x) + p(x)\,y'(x) + q(x)\,y(x) = 0.$$

3 If $x=a$ is a RSP of the DE then $(x-a)\,p(x)$ and $(x-a)^2\,q(x)$ must BOTH be analytic ('well-behaved') at $x=a$.

For the DE of interest

4 $x=0$ is an irregular singular point
& $x=1$ is a regular singular point

SECTION B

2. We have

$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i \nu t}\, dt \,. \qquad (1)$$

We can write (1) in the form

2

$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\cos(2\pi\nu t)\, dt - i\int_{-\infty}^{\infty} f(t)\sin(2\pi\nu t)\, dt \,. \qquad (2)$$

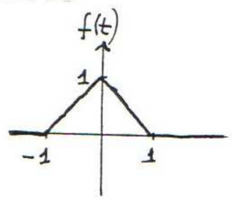
If $f(t)$ is an EVEN function then the second integral in (2)
3 is zero because the integrand is an ODD function of t. Hence
we can write (1 no explanation)

$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\cos(2\pi\nu t)\, dt \,. \qquad (3)$$

However, the integrand in (3) is an EVEN function of t &
2 we have

$$\mathcal{F}[f(t)] = 2\int_{0}^{\infty} f(t)\cos(2\pi\nu t)\, dt \,. \qquad (4)$$

(1 no explanation)



$$F(\nu) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i \nu t}\, dt \,. \qquad (5)$$

However, $f(t)$ is an EVEN function and we can use (4) to write

$$F(\nu) = 2\int_{0}^{\infty} f(t)\cos(2\pi\nu t)\, dt$$

$$= 2\int_{0}^{1} (1-t)\cos(2\pi\nu t)\, dt \qquad (6)$$

⇐ (4 marks)

Now

$$\int_{0}^{1}\cos(2\pi\nu t)\, dt = \frac{\sin(2\pi\nu)}{(2\pi\nu)} \,, \qquad (7)$$

(1) (3) (indefinite integrals 4 marks)

14

$$\int_{0}^{1} t\cos(2\pi\nu t)\, dt = \frac{\sin(2\pi\nu)}{2\pi\nu} + \frac{[\cos(2\pi\nu) - 1]}{(2\pi\nu)^2} \,. \qquad (8)$$

~~...~~ (messed up integration ...)

2. (cont). Hence, we obtain (-1 for no cos 2θ result)

$$F(\nu) = \frac{2\left[1-\cos(2\pi\nu)\right]}{(2\pi\nu)^2} = \left[\frac{\sin(\pi\nu)}{\pi\nu}\right]^2. \qquad (9)$$

The inverse Fourier transform gives

$$f(t) = \int_{-\infty}^{\infty} \frac{\sin^2(\pi\nu)}{(\pi\nu)^2} \cdot e^{2\pi i\nu t}\, d\nu, \qquad (2)$$

explain

$$= 2\int_0^{\infty} \cos(2\pi\nu t)\, \frac{\sin^2(\pi\nu)}{(\pi\nu)^2}\, d\nu. \qquad (2) \qquad (10)$$

9

When $t = 1/2$, $f(\frac{1}{2}) = \frac{1}{2}$, and (10) yields (3)

$$\frac{1}{2} = 2\int_0^{\infty} \cos(\pi\nu)\, \frac{\sin^2(\pi\nu)}{(\pi\nu)^2}\, d\nu. \qquad (11)$$

Finally, let $x = \pi\nu$ in (11):

$$\int_0^{\infty} \cos x \cdot \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{4}. \qquad (2)$$

SECTION B

3) First part is standard book-work. We can write

$$\text{grad}\,\psi = \sum_{i=1}^{3} (\text{grad}\,\psi)_i \, \underline{e}_i \, , \tag{1}$$

4 where $(\text{grad}\,\psi)_i = (\text{grad}\,\psi) \cdot \underline{e}_i = \dfrac{d\psi}{ds_i}$. (2)

For a small change $q_i \to q_i + dq_i$ along the ith coordinate line we have

2 $$d\psi = \left(\frac{\partial \psi}{\partial q_i}\right) dq_i \, , \tag{3}$$

2 $$ds_i = h_i \, dq_i \, .$$

Hence,

$$\frac{d\psi}{ds_i} = \frac{1}{h_i}\left(\frac{\partial \psi}{\partial q_i}\right) , \qquad (i = 1, 2, 3) \, . \tag{4}$$

2 We now have :

$$\text{grad}\,\psi = \sum_{i=1}^{3} \frac{1}{h_i}\left(\frac{\partial \psi}{\partial q_i}\right) \underline{e}_i \; \Big] \Leftarrow \tag{5}$$

For the given system we have

$$\underline{r}\,(q_1, q_2, q_3) = q_1 q_2 \cos q_3 \, \underline{i} + q_1 q_2 \sin q_3 \, \underline{j} + \tfrac{1}{2}(q_1^2 - q_2^2) \, \underline{k} \, .$$

Hence, we obtain

$$\left.\begin{aligned}
&2 \quad \left(\frac{\partial \underline{r}}{\partial q_1}\right) = q_2 \cos q_3 \, \underline{i} + q_2 \sin q_3 \, \underline{j} + q_1 \underline{k} \, , \\
&2 \quad \left(\frac{\partial \underline{r}}{\partial q_2}\right) = q_1 \cos q_3 \, \underline{i} + q_1 \sin q_3 \, \underline{j} - q_2 \underline{k} \, , \\
&2 \quad \left(\frac{\partial \underline{r}}{\partial q_3}\right) = -q_1 q_2 \sin q_3 \, \underline{i} + q_1 q_2 \cos q_3 \, \underline{j}
\end{aligned}\right] \tag{6}$$

From these results we find

6 $$h_1 = |\partial \underline{r}/\partial q_1| = (q_1^2 + q_2^2)^{\frac{1}{2}} \, , \quad h_3 = \left|\frac{\partial \underline{r}}{\partial q_3}\right| = q_1 q_2 \, . \tag{7}$$
$$h_2 = |\partial \underline{r}/\partial q_2| = (q_1^2 + q_2^2)^{\frac{1}{2}}$$

The unit base vector now follow from (6) & (7) :

$$\underline{e}_i = \frac{1}{h_i}\left(\frac{\partial \underline{r}}{\partial q_i}\right) \quad , (i = 1, 2, 3) . \qquad (8)$$

For the point of interest : $h_1 = h_2 = \sqrt{2}$, $h_3 = 1$.

We also find

8

$$\left.\begin{aligned}(\underline{e}_1)_P &= \tfrac{1}{2}\underline{i} + \tfrac{1}{2}\underline{j} + \tfrac{1}{\sqrt{2}}\underline{k} ,\\ (\underline{e}_2)_P &= \tfrac{1}{2}\underline{i} + \tfrac{1}{2}\underline{j} - \tfrac{1}{\sqrt{2}}\underline{k} ,\\ (\underline{e}_3)_P &= -\tfrac{1}{\sqrt{2}}\underline{i} + \tfrac{1}{\sqrt{2}}\underline{j}\end{aligned}\right]$$

$$\left(\frac{\partial \psi}{\partial q_1}\right)_P = \left(\frac{\partial \psi}{\partial q_2}\right)_P = \sqrt{2} \; , \quad \left(\frac{\partial \psi}{\partial q_3}\right) = -\sqrt{2} .$$

Finally , $(\text{grad}\,\psi)_P = 2\underline{i}$] $\Leftarrow$ (9)