King’s College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CM102D Mathematics for Science Students

Summer 2005

Time allowed: THREE Hours

Candidates should answer all SIX questions from SECTION A,
and no more than TWO questions from SECTION B.
No credit will be given for answering further questions.

The approximate mark for each part of a question is indicated in square brackets.

You must NOT use a calculator for this paper.

TURN OVER WHEN INSTRUCTED

SECTION A – Answer all parts of this section

1.1 Starting from the expression $e^{i\theta} = \cos\theta + i\sin\theta$, find an expression for $e^{4i\theta}$ in terms of $\cos\theta$ and $\sin\theta$. Use this to show that

$$\cos(4\theta) = 8\cos^4\theta - 8\cos^2\theta + 1$$

[7 marks]

1.2. Find an equation of the plane which passes through the points (1,1,0), (2,0,0) and (2,-1, -1), and write down the unit vector normal to that plane.

[7 marks]

1.3 Solve the following equation for real x:

$$\sinh x + 3\cosh x = 3$$

[7 marks]

1.4 Show that the expression

$$f(x, y) = x^2 - 2xy + 2y^2 - 6y + 3$$

has a stationary point at $(x, y) = (3,3)$. What type of stationary point is it?

[7 marks]

1.5 Show that $\sec x = 1 + \dfrac{x^2}{2} + \dfrac{5x^4}{24} + \cdots$ for small x.

[7 marks]

1.6 Evaluate $\lim\limits_{x\to 0} \dfrac{\sinh x - x}{\sin x - x}$.

[7 marks]

SECTION B - Answer TWO questions

2. Show that

$$\int \frac{x\,\mathrm{d}x}{(1-x^2)} = -\ln\sqrt{1-x^2} + c$$

where c is a constant of integration.

[5 marks]

Use a suitable substitution to show that $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \sin^{-1} x + c$

where c is a constant of integration.

[5 marks]

Use the method of integrating factors, and the results above, to solve the first order differential equation

$$(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x} - xy = 1$$

[15 marks]

If $y = \frac{\pi}{3}$ when $x = \frac{\sqrt{3}}{2}$, show that the full solution is

$$y = \frac{-\frac{\pi}{6} + \sin^{-1} x}{\sqrt{1-x^2}}$$

[5 marks]

3. Use integration by parts to prove that

$$\int \sin x \sinh x\,\mathrm{d}x = \tfrac{1}{2}(\sin x \cosh x - \sinh x \cos x) + c$$

where c is a constant of integration.

[10 marks]

Hence, show that the definite integral

$$\int_{-\pi/2}^{\pi/2} \int_{y=-x}^{x} e^{y} \sin x\,\mathrm{d}x\mathrm{d}y \quad = \quad 2\cosh\left(\frac{\pi}{2}\right).$$

[15 marks]

Explain how you could deduce that the value of the very similar integral

$$\int_{-\pi/2}^{\pi/2} \int_{y=-\infty}^{\infty} e^{y^2} \sin x\,\mathrm{d}x\mathrm{d}y$$

was zero without integrating at all.

[5 marks]

4. The matrix A is $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}$.

Show that the determinant of A is -3.

[6 marks]

The inverse of matrix A is $A^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ a & b & \frac{1}{3} \\ c & d & -\frac{1}{3} \end{pmatrix}$.

Determine the values of the elements a, b, c and d, and hence write down the matrix A^{-1}.

[9 marks]

Hence, or otherwise, solve the system of simultaneous equations:

$$\begin{aligned} x \quad\; + z &= 2 \\ y + z &= 1 \\ 2x + 2y + z &= 9 \end{aligned}$$

[7 marks]

Find the eigenvalues of matrix A.

[8 marks]