Candidate number:	Desk number:

King's College London

UNIVERSITY OF LONDON

This paper is part of an examination of the College counting towards the award of a degree. Examinations are governed by the College Regulations under the authority of the Academic Board.

B.Sc. EXAMINATION

4CCP1350 Mathematics for Physics I

Examiner: Dr Jean Alexandre

Summer 2010

Time allowed: THREE hours

Candidates may answer as many parts as they wish from SECTION A, but the total mark for this section will be capped at 40.

Candidates should answer no more than ONE question from SECTION B and no more than ONE question in section C. No credit will be be given for answering a further question from these sections.

The approximate mark for each part of a question is indicated in square brackets.

You may use a College-approved calculator for this paper. The approved calculator are the Casio fx83 and Casio fx85.

DO NOT REMOVE THIS EXAM PAPER FROM THE EXAMINATION ROOM

For examiner's use only

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TURN OVER WHEN INSTRUCTED

Throughout the paper, a dot over a letter corresponds to a time derivative.

SECTION A

Answer SECTION A on the question paper in the space below each question.
If you require more space, use an answer book.

Answer as many part of this section as you wish.
The final mark for this section will be capped at 40.

1.1 Calculate

$$\int_0^1 dx\ x \cosh x$$

[4 marks]

1.2 Define the angular momentum of a point particle, and state one situation where it is conserved. In this latter situation, explain why the trajectory is planar.

[4 marks]

1.3 Determine, giving your reasoning, whether the following integral converges or diverges

$$\int_0^\infty dx\, \frac{\cos x - 1}{x^{5/2}}$$

[6 marks]

1.4 The force applied on a point particle with electric charge q, moving in a magnetic field $\mathbf{B}$ with velocity $\mathbf{v}$, is $q\mathbf{v} \times \mathbf{B}$. Show that this magnetic force does no work, and explain its effect on the motion of the charged particle.

[4 marks]

1.5 Derive an equation for the plane containing the point with coordinates $(1, 2, 3)$, and perpendicular to the vector $(-1, 1, -2)$.

[4 marks]

1.6 Derive an expression for the moment of inertia of a solid homogeneous cylinder, with respect to its axis of symmetry, in terms of its radius and mass.

[6 marks]

1.7 Solve the equation $z^3 = 8$ for the complex number z.

[4 marks]

1.8 Explain how the characteristic features of a mechanical resonance change, when the quality factor of the oscillating system increases. Describe the effect on the speed and the phase of the oscillations.

[3 marks]

1.9 Show that

$$(\cos x)^3 \sin x = \frac{1}{8}\sin(4x) + \frac{1}{4}\sin(2x)$$

[5 marks]

1.10 The moment of inertia of a solid homogeneous ball or radius R and mass M, with respect to an axis of symmetry, is $\frac{2}{5}MR^2$. If this ball rolls with speed V without slipping, explain why its kinetic energy is

$$E_k = \frac{7}{10}MV^2$$

[4 marks]

1.11 Solve the differential equation $y' + xy = 0$, for $y(0) = 1$, where a prime denotes a derivative with respect to x.

[5 marks]

1.12 Explain why, when a rigid body rolls without slipping on a planar surface, contact forces are not dissipative.

[3 marks]

1.13 Solve for x the equation $2^x = e^{x^3}$.

[4 marks]

1.14 Two points particles A_1 and A_2, with masses m_1 and m_2 respectively, interact gravitationnaly with each other. Define the reduced mass of the system (A_1, A_2) and write Newton's second law in terms of the equivalent one-body problem.

[4 marks]

SECTION B - Answer ONE question

Answer section B in an answer book

2 The aim of this exercise is to study some properties of the matrix

$$\mathbf{M} = \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$$

2.a) Calculate the determinant and the inverse of $\mathbf{M}$.

[5 marks]

2.b) Calculate the eigenvalues of $\mathbf{M}$, and find a set of corresponding eigenvectors.

[8 marks]

2.c) Consider the following matrix

$$\mathbf{S} = \begin{pmatrix} 5/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 7/4 \end{pmatrix},$$

and the orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2)$ defined by

$$\mathbf{e}_1 = (1/2)\mathbf{i} + (\sqrt{3}/2)\mathbf{j} \qquad \mathbf{e}_2 = (-\sqrt{3}/2)\mathbf{i} + (1/2)\mathbf{j}$$

Consider the action of $\mathbf{S}$ on a vector $\mathbf{u}$ written in the form $\mathbf{u} = a\ \mathbf{e}_1 + b\ \mathbf{e}_2$, and show that $\mathbf{S}$ represents a scaling of factor 2 in the direction of $\mathbf{e}_1$.
Using this result, give without calculation the eigenvalues and corresponding eigenvectors for $\mathbf{S}$.

[6 marks]

2.d) Consider the following matrix

$$\mathbf{R} = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix},$$

and the orthonormal basis $(\mathbf{e}_3, \mathbf{e}_4)$ defined by

$$\mathbf{e}_3 = (\sqrt{3}/2)\mathbf{i} + (1/2)\mathbf{j} \qquad \mathbf{e}_4 = (-1/2)\mathbf{i} + (\sqrt{3}/2)\mathbf{j}$$

Consider the action of $\mathbf{R}$ on a vector $\mathbf{u}$ written in the form $\mathbf{u} = c\ \mathbf{e}_3 + d\ \mathbf{e}_4$, and show that $\mathbf{R}$ represents a reflection in the direction of $\mathbf{e}_3$.
Using this result, give without calculation the eigenvalues and corresponding eigenvectors for $\mathbf{R}$.

[6 marks]

2.e) Show that $\mathbf{M}$ represents the composition of the scaling of question **c)**, followed by the reflection of question **d)**.

[5 marks]

SEE NEXT PAGE

3 The aim of this exercise is to study some properties of the Gamma function, defined for real $x > 0$ by

$$\Gamma(x) = \int_0^\infty dt\ t^{x-1}\ e^{-t}$$

3.a) Justify that the integral $\Gamma(x)$ is convergent for $x > 0$, and calculate $\Gamma(1)$.

[5 marks]

3.b) Using an integration by parts, show that $\Gamma(x+1) = x\Gamma(x)$.

[5 marks]

3.c) Infer from the previous part of this question that, if n is a natural number, then

$$\Gamma(n+1) = n!$$

[5 marks]

3.d) You may assume that the differentiation with respect to x can be done though the integral over t. Show that

$$\frac{d\Gamma(x)}{dx} = \int_0^\infty dt\ \ln(t)\ t^{x-1}\ e^{-t}$$

[5 marks]

3.e) Show that

$$\Gamma(1/2) = 2\int_0^\infty du\ e^{-u^2}$$

[5 marks]

3.f) By writing

$$\Big(\Gamma(1/2)\Big)^2 = 4\int_0^\infty du \int_0^\infty dv\ e^{-u^2-v^2},$$

and, using a change of variable to polar coordinates, prove that

$$\Gamma(1/2) = \sqrt{\pi}.$$

[5 marks]

SECTION C - Answer ONE question

Answer section C in an answer book

4.a) Show that the equation describing a simple pendulum of legnth l, in a uniform and constant gravitational field g, without friction, is

$$\ddot{\theta} + \omega^2 \sin\theta = 0,$$

where $\omega = \sqrt{g/l}$ and θ is the angle the pendulum makes with the vertical.

[6 marks]

4.b) Give the general solution of this differential equation for small angles, when one can assume that $\sin\theta \simeq \theta$, and give an expression for the period of oscillations T_0.

[4 marks]

4.c) If one doesn't assume small angles, show that

$$(\dot{\theta})^2 = 2\omega^2(\cos\theta - \cos\theta_0),$$

where θ_0 is the maximum amplitude for the angle θ.

[5 marks]

4.d) Show that the period of oscillations is given by

$$T = \frac{2\sqrt{2}}{\omega} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}},$$

[5 marks]

4.e) Using a Taylor expansion for small θ_0, show that

$$T = T_0 \left(1 + \frac{\theta_0^2}{16}\right) + \mathcal{O}(\theta_0^4).$$

[10 marks]

5 A satellite of mass m is orbiting the Earth, on a circular trajectory of radius R.

5.a) Show that the speed of the satellite is

$$V_{circ} = \sqrt{\frac{MG}{R}},$$

where M is the Earth mass and G is the gravitational constant.

[6 marks]

5.b) A circular trajectory, of radius R_{geo}, is geostationary if the angular velocity on this trajectory is equal to the Earth's angular velocity Ω. In this case, show that

$$R_{geo} = \left(\frac{MG}{\Omega^2}\right)^{1/3},$$

and give the value of R_{geo}, assuming $M = 5.97 \times 10^{24}$kg and $G = 6.67 \times 10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2}$.

[6 marks]

5.c) Using polar coordinates (r, θ) defined from the centre of the Earth, show that the expressions for the energy E and the amplitude of the angular momentum $\mathcal{L}$ on a conic trajectory are

$$E = \frac{1}{2}m(\dot{r})^2 - \frac{mMG}{r} + \frac{\mathcal{L}^2}{2mr^2} \qquad \mathcal{L} = mr^2\dot{\theta}$$

[6 marks]

5.d) Infer from part 4.c) that the energy and the amplitude of the angular momentum on the geostationary trajectory are

$$E_{geo} = -\frac{mMG}{2R_{geo}} \qquad \mathcal{L}_{geo} = m\sqrt{MGR_{geo}}$$

[6 marks]

5.e) From the geostationary trajectory, one wishes to send the satellite on a parabolic trajectory, tangential to the circular trajectory, by switching on its engine for a short time. Show that the initial speed one needs to give to the satellite is

$$V_{para} = \sqrt{2}V_{geo},$$

where V_{geo} is the speed on the geostationary trajectory, and give the value of V_{para}.

[6 marks]

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