Candidate number:	Desk number:

King's College London

UNIVERSITY OF LONDON

This paper is part of an examination of the College counting towards the award of a degree. Examinations are governed by the College Regulations under the authority of the Academic Board.

B.Sc. EXAMINATION

4CCP1350 Mathematics and Mechanics for Physics I

Examiner: Dr Jean Alexandre

Summer 2009

Time allowed: THREE hours

Candidates may answer as many parts as they wish from SECTION A, but the total mark for this section will be capped at 40.

Candidates should answer no more than TWO questions from SECTION B.
No credit will be be given for answering a further question from this section.

The approximate mark for each part of a question is indicated in square brackets.

You may use a College-approved calculator for this paper.
The approved calculator are the Casio fx83 and Casio fx85.

DO NOT REMOVE THIS EXAM PAPER
FROM THE EXAMINATION ROOM

For examiner's use only

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TURN OVER WHEN INSTRUCTED

Throughout the paper, a dot over a letter corresponds to a time derivative.

SECTION A

Answer SECTION A on the question paper in the space below each question. If you require more space, use an answer book.

Answer as many part of this section as you wish. The final mark for this section will be capped at 40.

1.1 Calculate

$$\int_0^1 dx\ xe^x$$

[4 marks]

1.2 Define an inertial frame.

[3 marks]

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1.3 Explain wether the following integrals converge or diverge

$$\int_0^\infty dx\,\frac{\cos x}{x^2} \qquad\qquad \int_0^\infty dx\,\frac{\sqrt{x}}{\sinh x}$$

[4 marks]

1.4 A car is moving in a straight line, with a constant acceleration of 5 ms^{-2}. Calculate the angle a pendulum in the car makes with the vertical (the acceleration due to gravity is $g = 9.8$ ms^{-2}).

[5 marks]

1.5 In two dimensions, the rotation by the angle $\pi/3$ is represented by the matrix

$$\begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$

The vector $\mathbf{u} = (1, 3)$ is rotated by the angle $\pi/3$ into $\mathbf{u}'$. Calculate the coordinates of $\mathbf{u}'$ and check that $|\mathbf{u}'| = |\mathbf{u}|$.

[5 marks]

1.6 Compute the power which is required to stop a 10^4 kg truck with speed 90 kmh^{-1}, in 10 seconds.

[3 marks]

1.7 Solve the equation $z^3 = 1$, where z is a complex number.

[4 marks]

1.8 State the parallel axis theorem.

[3 marks]

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1.9 Using complex numbers, show that

$$(\sin x)^3 = \frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x)$$

[5 marks]

1.10 Show that the angular momentum of a point particle subjected to a central force is conserved.

[5 marks]

1.11 Prove that, for any real x, $\cosh^2 x - \sinh^2 x = 1$.

[5 marks]

1.12 Solve the equation $\ddot{x} + \omega^2 x = 0$ when $x(0) = a$ and $\dot{x}(0) = 0$.

[5 marks]

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1.13 Derive the equation for the plane perpendicular to the vector $\mathbf{u} = (1, 2, 3)$, that contains the point A of coordinates $(3, 2, 1)$.

[4 marks]

1.14 A solid homogeneous cylinder of mass M, radius R and height H revolves about its axis of cylindrical symmetry. Derive an expression for its moment of inertia about this axis.

[5 marks]

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SECTION B - Answer TWO questions
Answer section B in an answer book

2. The aim of this question is to study, in two dimensions, some properties of a linear transformation obtained by combining a rotation and a projection.

a) A rotation of the basis vectors $\mathbf{i}, \mathbf{j}$ by the angle $\pi/6$ gives the vectors $\mathbf{i}', \mathbf{j}'$. Express $\mathbf{i}', \mathbf{j}'$ in terms of $\mathbf{i}$ and $\mathbf{j}$, and infer that the matrix representing this rotation is

$$\mathbf{R} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}.$$

(Hint: $\sin(\pi/6) = 1/2$). Calculate the determinant of $\mathbf{R}$. [8 marks]

b) The matrix representing the projection onto the straight line given by $y = 2x$ is

$$\mathbf{P} = \begin{pmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{pmatrix}$$

Calculate $\det(\mathbf{P})$ and $\mathbf{P}^2$, and explain the significance of these results in terms of the mapping provided by the projection. [7 marks]

c) Justify that the matrix $\mathbf{M} = \mathbf{RP} - \mathbf{PR}$ has an inverse. Calculate its inverse and show that

$$\mathbf{M}^{-1} = 4\mathbf{M}$$

[8 marks]

d) Show that $\mathbf{M}^2$ represents a scaling, and give the corresponding scale factor. [7 marks]

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3. The equation for the elliptical trajectory of the Earth orbiting the Sun is

$$r = \frac{R}{1 + a\cos\theta},$$

where (r, θ) are the polar coordinates centered at the Sun, and the constants R and $a < 1$ are to be determined.

a) Show that the energy E of the Earth and the projection of its the angular momentum $\mathcal{L}$ on the axis perpendicular to the trajectory are

$$\begin{aligned} E &= \frac{1}{2}m\Big((\dot{r})^2 + (r\dot{\theta})^2\Big) - \frac{GMm}{r} \\ \mathcal{L} &= mr^2\dot{\theta} \end{aligned}$$

where m and M are respectively the masses of the Earth and the Sun. [6 marks]

b) Show that, at any point of the trajectory,

$$\begin{aligned} \dot{r} &= \frac{\mathcal{L}}{mR}a\sin\theta \\ r\dot{\theta} &= \frac{\mathcal{L}}{mR}(1 + a\cos\theta) \end{aligned}$$

[8 marks]

c) Show that the energy of the Earth can be written

$$E = \frac{m}{2}\left(\frac{\mathcal{L}}{mR}\right)^2(1 + a^2) - \frac{GMm}{R} + ma\cos\theta\left[\left(\frac{\mathcal{L}}{mR}\right)^2 - \frac{GM}{R}\right],$$

and conclude that

$$R = \frac{\mathcal{L}^2}{GMm^2}$$

[8 marks]

d) Give an expression for the eccentricity a in terms of the energy, and conclude that

$$|E| \leq \frac{GMm}{2R}$$

[8 marks]

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4. If x is a real argument, consider the differential equation (E)

$$y' + \left[\frac{x^2 - x^3 - 1}{x^2(1-x^2)}\right] y = \frac{1}{x^2\sqrt{1-x^2}},$$

where the function y is to be determined, and y' is its derivative with respect to x.

a) State for what range of values of x the equation (E) is defined, and show that

$$\frac{x^2 - x^3 - 1}{x^2(1-x^2)} = \frac{d}{dx}\left(\frac{1}{x} + \frac{1}{2}\ln(1-x^2)\right)$$

[4 marks]

b) Show that the general solution of the homogeneous differential equation associated with (E) is

$$y_h(x) = \frac{y_0 e^{-1/x}}{\sqrt{1-x^2}},$$

where y_0 is a constant. [8 marks]

c) Give the limit of $y_h(x)$ when $x \to 0$, with $x > 0$. [4 marks]

d) Using the variation of parameter method, show that the general solution of the equation (E) is

$$y = \frac{y_0 e^{-1/x} - 1}{\sqrt{1-x^2}}$$

[8 marks]

e) Show that, if $y_0 = e$, the solution of question **d)** is finite when $x \to 1$. [6 marks]

5. The aim of this question is to study the effect of the Coriolis force on the trajectory of a ball bowled down a bowling alley. The bowling venue, located at the latitude λ and with bowling lanes in the direction South-North, is a non-inertial frame, due to the Earth spinning with angular velocity Ω. In what follows, the x-axis is in the direction West-East, with its basis vector $\mathbf{i}$ towards the East; the y-axis is in the direction of the lanes, with its basis vector $\mathbf{j}$ towards the North; the z-axis is vertical, making the angle λ with the equatorial plane. The bowling ball, of mass m, is thrown from the origin of the coordinates, with initial velocity $v_0\mathbf{j}$, and the Coriolis force is $\mathbf{f} = 2m\Omega\mathbf{v} \times \mathbf{n}$, where $\mathbf{v}$ is the velocity of the bowling ball, and $\mathbf{n}$ is the unit vector along the polar axis of the Earth.

a) Write the equations of motion in the horizontal plane (x, y), and neglecting $\dot{z}$. Perform a first integration, taking into account the initial conditions, and show that, if $x\Omega << v_0$,

$$\dot{x} = 2\Omega y \sin\lambda \qquad \text{and} \qquad \dot{y} = v_0$$

[7 marks]

b) Show that the deviation of the trajectory from the line $x = 0$ is towards the East and is equal to

$$\delta(t) = \Omega v_0 t^2 \sin\lambda$$

Explain whether the effect of the Coriolis force is more or less important at the North Pole than on the Equator. [6 marks]

c) The length of the bowling lanes is L. Show that the deviation from the line $x = 0$, at the end of the trajectory, is

$$\delta_{end} = \Omega\frac{L^2}{v_0}\sin\lambda$$

The bowling venue is located in London, with latitude 50°, the length of the bowling lanes is 20 m, and the Earth makes a complete rotation in 24 hours. Calculate the deviation δ_{end} for a bowling ball thrown with speed 5 ms^{-1}, and discuss the significance of the result.

[7 marks]

d) The equation of motion along the z-axis has been neglected. Discuss its relevance.

[4 marks]

e) Show that, if $\Omega L << v_0$ and if the ball does not slip, the kinetic energy of the ball at the end of the trajectory is

$$E_k = \frac{7}{10}mv_0^2$$

[6 marks]

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