King's College London

UNIVERSITY OF LONDON

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B.Sc. EXAMINATION

4CCP1350 / CP1350 Mathematics and Mechanics for Physics I

Examiner: Professor Alison Mainwood and Dr Jean Alexandre

Summer 2008

Time allowed: THREE Hours

Answer SECTION A on the question paper.
Candidates may answer as many parts as they wish from SECTION A, but the total mark for this section will be capped at 40.

Candidates should answer no more than TWO questions from SECTION B.
No credit will be given for answering a further question from this section.

The approximate mark for each part of a question is indicated in square brackets.

You may use a College-approved calculator for this paper. The approved calculators are the Casio fx83 and Casio fx85.

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Throughout this paper, t denotes the time, and a dot over a letter denotes a time derivative. $\vec{e}_r, \vec{e}_\theta$ are the usual orthogonal basis vectors in polar coordinates, with $\dot{\vec{e}}_r = \dot{\theta}\vec{e}_\theta,\ \dot{\vec{e}}_\theta = -\dot{\theta}\vec{e}_r$.

SECTION A

Answer as many parts of this section as you wish.

The final mark for this section will be capped at 40.

1.1) Solve the following equation for real x:

$$\sinh x + \cosh x = 3.$$

[3 marks]

1.2) Show that for any two complex numbers, z_1 and z_2, $|z_1||z_2| = |z_1 z_2|$.

[3 marks]

1.3) Explain why the following matrix has no inverse.

$$\begin{pmatrix} x^2 & x \\ xy & y \end{pmatrix}$$

[4 marks]

1.4) Use De Moivre's Theorem to find an expression for $\cos 3\theta$ in terms of $\cos\theta$.

[6 marks]

1.5) Define an inertial frame.

[2 marks]

1.6) Derive a relation between the power of a force acting on a particle and the kinetic energy of this particle.

[3 marks]

1.7) Give an expression for the Coriolis force, and explain its origin.

[5 marks]

1.8) A car is moving with a constant acceleration of 4 m.s^{-2}. Calculate the angle which a simple pendulum in the car makes with respect to the vertical (the acceleration due to gravity is $g \simeq 9.8$ m.s^{-2}).

[4 marks]

1.9) A comet is travelling on an elliptical trajectory. At what points, on this trajectory, is the comet's velocity perpendicular to its acceleration?

[4 marks]

1.10) A point particle moves with constant angular velocity ω on a circular trajectory of radius R. Derive expressions for its velocity $\vec{v}$ and acceleration $\vec{a}$.

[5 marks]

1.11) What is a central force? State which quantity is conserved in motion generated by such a force. Describe the trajectory of a point particle travelling under the influence of a central force.

[4 marks]

1.12) Write down the general forms of the different Kepler trajectories. Give the sign of the energies of particles travelling on each trajectory (assuming that the potential vanishes at infinity).

[4 marks]

1.13) Three solid balls are arranged on a straight line. The first, of mass 1kg, is at the origin ($x_1 = 0$), the second, of mass 2kg, is located 1 meter away ($x_2 = 1$m) and the third, of mass 3kg, is 2 meters away from the origin ($x_3 = 2$m). Calculate the coordinate x_C of the centre of mass of the system.

[4 marks]

1.14) A rigid homogeneous disc of mass M and radius R revolves about its axis of symmetry perpendicular to the disc. Give an expression for its moment of inertia about this axis.

[3 marks]

1.15) A rigid homogeneous ball of mass M and radius R rolls downhill without slipping; its centre of mass has velocity $\vec{v}$. The moment of inertia of the ball with respect to an axis of symmetry is $\frac{2}{5}MR^2$. Show that the kinetic energy of the ball is $\frac{7}{10}Mv^2$.

[6 marks]

SECTION B – Answer TWO questions

2)

Find the particular solution of the first order differential equation

$$\tan x \frac{dy}{dx} + y = \sin x \tan x$$

if y is finite when $x = 0$.

[22 marks]

Differentiate the particular solution to demonstrate that it satisfies the original differential equation.

[8 marks]

[Hint: The integrating factor of a differential equation of the form

$$\frac{dy}{dx} + yP(x) = Q(x)$$

is given by $\exp(\int P(x)dx)$]

3) A point mass m moving along the x-axis can oscillate under the influence of the restoring force $-kx$. These oscillations are damped, due to the friction force $-\lambda\dot{x}$ (k and λ are positive constants).

a) Show that the equation of motion can be written as

$$\ddot{x} + 2\frac{\dot{x}}{\tau} + \omega^2 x = 0,$$

and give expressions for τ and ω in terms of m, k, λ.

[5 marks]

b) If $\omega\tau > 1$ (weak damping), show that the general solution of the equation of motion is

$$x(t) = C \exp\left(-\frac{t}{\tau}\right) \cos\left(\Omega t + \phi\right),$$

where C, ϕ are constants of integration and $\Omega = \sqrt{\omega^2 - 1/\tau^2}$.

[7 marks]

c) Derive an expression for the specific solution corresponding to the initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$.

[5 marks]

d) When $\omega\tau >> 1$ show that the total energy of the oscillator is

$$E \simeq \frac{k^2}{2} x_0^2 \exp\left(-\frac{2t}{\tau}\right).$$

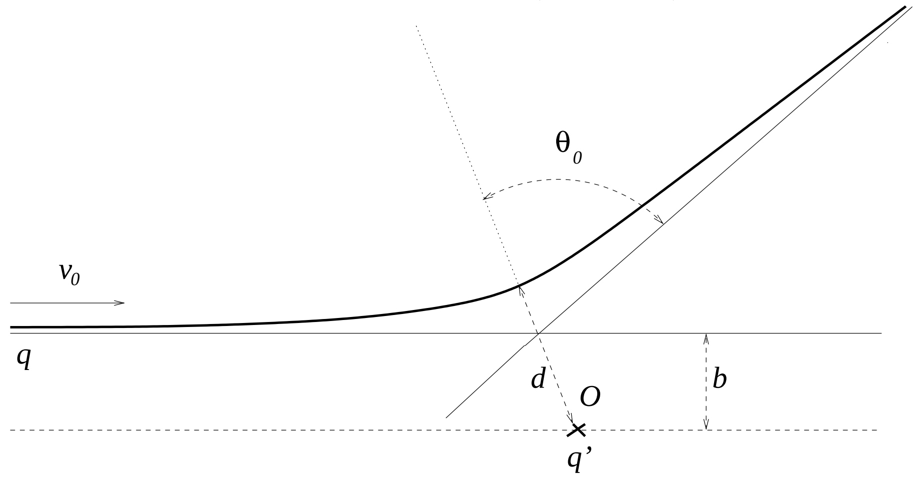
[8 marks]

e) Show that the rate of change of energy during the time $\Delta t << \tau$ is

$$\left|\frac{\Delta E}{E}\right| \simeq 2\frac{\Delta t}{\tau}.$$

[5 marks]

4) An electron of mass m and charge q is scattered by a (negative) fixed charge q', centred at the origin of the coordinates O. The trajectory is hyperbolic, and the initial velocity of the electron is $\vec{v}_0$, along one of the asymptotes of the trajectory, which is at a distance b from O (see figure).



a) Show that the angular momentum $\vec{\mathcal{L}}$ is conserved and, using polar coordinates centred on O, show that $\vec{\mathcal{L}} = -mbv_0\vec{k}$, where $\vec{k}$ is the unit vector perpendicular to plane of the trajectory.

[8 marks]

b) Show that the total energy of the electron is given by

$$E = \frac{1}{2}m(\dot{r})^2 + \frac{1}{2}m\frac{b^2v_0^2}{r^2} - \frac{qq'}{4\pi\epsilon_0 r}.$$

[5 marks]

c) Show that the distance of closest approach between the electron and the fixed charge is $d = \sqrt{a^2+b^2} - a$, where $a = qq'/(4\pi\epsilon_0 m v_0^2)$.

[7 marks]

d) The equation of the trajectory, in polar coordinates, can be written as

$$r = R/(e\cos\theta - 1),$$

where R and e depend on the parameters of the problem and $e > 1$. Show that the distance d calculated in c) satisfies

$$R\cos\theta_0 = d(1-\cos\theta_0),$$

where θ_0 is half the largest angle between the asymptotes, as shown on the figure.

[7 marks]

e) If the charge q' is positive instead, sketch the hyperbolic trajectory together with the centre of the force, showing the asymptotes.

[3 marks]

5) A majorette throws a stick of mass m and length l vertically, so that the centre of mass of the stick reaches a height h. In this question, heights are measured from the hand of the majorette. The centre of mass of the stick has initial speed v_0, and its initial angular velocity in its centre of mass frame is ω_0.

a) The stick can be assumed to be a homogeneous rod. Show that its moment of inertia, with respect to the axis of rotation perpendicular to the rod and passing through its centre of mass is

$$I = \frac{1}{12} m l^2 .$$

[5 marks]

b) Show that the total energy of the stick is

$$E = \frac{1}{2} m (\dot{z})^2 + \frac{1}{24} m l^2 \omega^2 + mgz,$$

where z and $\dot{z}$ are the height and the speed, respectively, of its centre of mass, and ω is its angular velocity in the centre of mass frame.

[5 marks]

c) If air friction can be neglected, explain why the angular velocity of the stick is conserved as the stick travels upwards, and show that

$$h = \frac{v_0^2}{2g} .$$

[7 marks]

d) Air friction induces a torque $-\lambda\omega$ in the centre of mass frame of the stick, where λ is a constant. Show that in this case the angular velocity is given by

$$\omega(t) = \omega_0 \exp\left(-\frac{\lambda t}{I}\right) .$$

[8 marks]

e) Assuming that friction does not affect the translational motion, show that, when the stick is caught by the majorette, its angular velocity is

$$\omega_1 = \omega_0 \exp\left(-\frac{2\lambda v_0}{gI}\right) .$$

[5 marks]