King’s College London

UNIVERSITY OF LONDON

This paper is part of an examination of the College counting towards the award of a degree. Examinations are governed by the College Regulations under the authority of the Academic Board.

B.Sc. EXAMINATION

CM102D Mathematics for Science Students

Examiner: Professor Alison Mainwood

Summer 2007

Time allowed: THREE Hours

Candidates may answer as many parts as they wish from SECTION A, but the total mark from this section will be capped at 40.

Candidates should answer no more than TWO questions from SECTION B. No credit will be given for answering further questions.

The approximate mark for each part of a question is indicated in square brackets.

You may NOT use a calculator for this paper.

TURN OVER WHEN INSTRUCTED

SECTION A

Answer as many parts of this section as you wish
Marks for this section will be capped at 40.

1.1) Evaluate the integral $\int_{y=-1}^{1}\int_{x=y}^{1}\left(x^2+y^2\right)\mathrm{d}x\mathrm{d}y$.

[4 marks]

1.2) Find the following limit.

$$\lim_{x\to 0}\frac{\sin x}{\sinh x}.$$

[4 marks]

1.3) Sketch the curve on an Argand diagram that satisfies the equation

$$|z|=4.$$

[4 marks]

1.4) Solve the following equation:

$$3\sinh x+4\cosh x=5$$

[5 marks]

1.5) If $V=(x^2-y^2)$, prove that $y\dfrac{\partial V}{\partial x}+x\dfrac{\partial V}{\partial y}=0$

[4 marks]

1.6) Expand $\tan x$ in a series, up to terms in x^3.

[6 marks]

1.7) Find the values of k such that the following equations are consistent:

$$x+(k+1)y+1=0$$
$$2kx+5y-3=0$$
$$3x+7y+1=0$$

[6 marks]

1.8) Solve the following differential equation, with the condition that $y=0$ when $x=0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=x+xy$$

[6 marks]

1.9) Find an equation of the plane which passes through the following points: (1,1,0), (2,0,0), (0,1,2)

[5 marks]

1.10) Matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$ has an inverse, $\boldsymbol{A}^{-1} = \begin{pmatrix} a & -1 & b \\ 0 & 1 & 0 \\ -\frac{1}{2} & -1 & c \end{pmatrix}$. Find the values of *a, b* and *c*.

[6 marks]

1.11) Evaluate the integral

$$\int_0^{\pi/4} \tan x \, \mathrm{d}x \, .$$

[5 marks]

1.12) Show that the function

$$f(x, y) = x^2 + 2xy - y^2 + 4y + 3$$

has a stationary point at $(x, y) = (-1, 1)$. What type of stationary point is it?

[5 marks]

SECTION B - Answer TWO questions

2) I_n is defined to be $\int x^n e^{-x^2} \mathrm{d}x$.

a) Calculate $I_1 \equiv \int xe^{-x^2} dx$.

[2 marks]

b) Use integration by parts to show that

$$I_n \equiv \int x^n e^{-x^2} \mathrm{d}x = \tfrac{1}{2}\left[(n-1)I_{n-2} - x^{n-1}e^{-x^2}\right]$$

[6 marks]

c) Show explicitly for the cases n = 1, 3, and 5 (or prove by induction) that in the case where n is an odd, positive integer, the definite integral

$$\int_0^\infty x^n e^{-x^2} dx = \frac{m!}{2}$$

where $m = \dfrac{(n-1)}{2}$.

[10 marks]

d) Given that $\int_0^\infty e^{-x^2} \mathrm{d}x = \tfrac{1}{2}\sqrt{\pi}$, find an arithmetic expression for $\int_0^\infty x^8 e^{-x^2} \mathrm{d}x$.

Hence, suggest a general solution for $\int_0^\infty x^n e^{-x^2} dx$ where n is an even, positive integer.

[12 marks]

3) a) Show that the indefinite integral $\int \cot x \, \mathrm{d}x = \ln|\sin x| + c$.

[5 marks]

b) Use the integrating factor method to find the particular solution of the first order differential equation

$$\sin x \frac{\mathrm{d}y}{\mathrm{d}x} + y\cos x = \cos x$$

if $y = 0$ when $x = \frac{\pi}{2}$.

[9 marks]

c) Find the particular solution of the similar differential equation,

$$\sin x \frac{\mathrm{d}y}{\mathrm{d}x} - y\cos x = \cos x$$

in the case where $y = 0$ when $x = \frac{\pi}{2}$.

[11 marks]

d) Sketch your solutions to parts b) and c) in the range $-\pi \le x \le \pi$.

[5 marks]

4) The three Pauli spin matrices, used in quantum mechanics, are defined as:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a) Show that $\sigma_1\sigma_2 = -\sigma_2\sigma_1 = i\sigma_3$, and derive similar expressions for $\sigma_2\sigma_3$ and $\sigma_3\sigma_1$.

[9 marks]

b) Determine the inverse matrices σ_1^{-1}, σ_2^{-1}, and σ_3^{-1}.

[8 marks]

c) Find the eigenvalues of the three Pauli spin matrices.

[8 marks]

d) Show that a general 2×2 matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ may be expressed as a linear combination of the Pauli matrices, $A\sigma_1 + B\sigma_1 + C\sigma_3 + DI$ for a suitable choice of scalars, *A, B, C, D*, where *I* is the 2×2 identity matrix.

[5 marks]

FINAL PAGE