BSc/MSci EXAMINATION

LOGIC

(CM328X Logic, 6CCM328A Logic)

MAY-JUNE 2010

TIME ALLOWED: TWO HOURS

THIS PAPER CONSISTS OF TWO SECTIONS, SECTION A AND SECTION B.

SECTION A CONTRIBUTES HALF THE TOTAL MARKS FOR THE PAPER. ANSWER ALL QUESTIONS IN SECTION A.

QUESTIONS IN SECTION B CARRY EQUAL MARKS, BUT IF MORE THAN TWO ARE ATTEMPTED THEN ONLY THE BEST TWO WILL COUNT.

QUESTIONS SHOULD BE ANSWERED IN THE SPACES PROVIDED ON THE QUESTION PAPER.

YOU HAVE BEEN PROVIDED A BOOKLET FOR ROUGH WORK.
IT MUST BE ATTACHED TO THIS PAPER, BUT WILL NOT BE MARKED.

NO CALCULATORS ARE PERMITTED.

SECTION A

QUESTION 1 **(10 points)**

(a) Express each of the following in the notation of propositional logic. ***Advice*****: If in doubt leave blank, as each incorrect answer loses as much as a correct answer gains.**

(i) if *p* then *q*, (ii) *p* if *q*, (iii) *p* only if *q*, (iv) only if *p* do we have *q*, (v) *p* provided that *q*, (vi) *p* unless *q*, (vii) *p* is a necessary and sufficient condition for *q*, (viii) *p* precisely if *q*, (ix) *p* but *q*, (x) neither *p* nor *q*.

(b) Use successive transformations to put the following formula into disjunctive normal form: $p\to(\neg q\wedge\neg(r\wedge\neg s))$**.**

(c) Express the same formula $p\to(\neg q\wedge\neg(r\wedge\neg s))$ **in prefix (Polish) notation.**

QUESTION 2 **(10 points)**

(a) Outline a proof that every substitution instance of a tautology is a tautology.

(b) Without writing out a proof, identify any redundant letters in the formula $[(p\vee\neg q)\wedge(r\wedge p)]\vee(r\wedge\neg p)$ and express it in least letter-set form.

(c) Construct a semantic decomposition tree to determine whether or not the propositional formula $[(p\to\neg q)\wedge(p\wedge r)]\vee[p\vee(q\wedge\neg r)]$ is a tautology.

QUESTION 3 **(10 points)**

Let α be the formula $\exists y[Py \wedge \forall y(Szx)] \rightarrow \exists z[\exists x(Rxy) \wedge \forall w(z=f(w))]$.

(a) Mark the free occurrences of x,y,z,w in α.

(b) Which of the following six substitutions are clean? $\alpha[x/x]$, $\alpha[y/x]$, $\alpha[z/x]$, $\alpha[g(y,a)/z]$, $\alpha[f(w)/y]$, $\alpha[x/y]$. *Advice* : Answer yes or no in each case; but if in doubt leave blank, as each incorrect answer loses as much as a correct answer gains.

(c) Define the concept of an *x*-variant of an interpretation for quantificational logic.

(d) Construct the finite transform of the following formula in a domain $D = \{1,2\}$ of two individuals: $\forall x[\exists y(Rxy) \rightarrow Px]$.

QUESTION 4 (10 points)

Express the following statements in the language of quantificational logic with identity. In each case specify a domain of discourse and a dictionary for all constants, function letters, and predicate letters employed.

(a) Any two sets that have exactly the same elements are identical.

(b) Every set has an element with which it shares no elements.

(c) For any rational number greater than zero there is a smaller one greater than zero.

(d) If someone wins then everyone else will congratulate him.

(e) There is a problem that can be solved only if no problem can be solved.

QUESTION 5 (10 points)

(a) Give (without calculations) an interpretation in a small finite domain that shows that $\exists x \exists y(Py \rightarrow Qx)$ does not logically imply $\exists y(Py) \rightarrow \exists x(Qx)$.

(b) Show by natural deduction that $\forall x[\exists y(Py) \rightarrow \forall z(Rxz)] \vdash \exists y(Py) \rightarrow \forall x(Rxx)$.

SECTION B

QUESTION 6 **(25 points)**

(a) What does it mean to say that a set of connectives for propositional logic is functionally complete?

(b) Explain briefly why the set {¬,∧,∨} is functionally complete.

(c) Using (b), show that the set {¬,→} is functionally complete.

(d) Outline a proof that the set {∧,∨,→,↔} is not functionally complete.

(e) Outline a proof that the set {¬,↔} is not functionally complete.

QUESTION 7 **(25 points)**

(a) State the least letter-set, finest splitting, interpolation and compactness theorems for propositional logic.

(b) Outline a direct semantic proof of the compactness theorem for propositional logic.

QUESTION 8

(25 points)

(a) What is prenex normal form? What is the prenex normal form theorem?

(b) Let α be the formula $\exists x(Px) \rightarrow \exists x(Qx)$. Get α into prenex normal form by a succession of transformations, using relabeling of bound variables, connective translation, quantifier interchange, and vacuous quantification.

(c) Use natural deduction centering on proof by contradiction to show that $\forall x \forall y \forall z(Rxy \rightarrow \neg Rxz) \vdash \neg \exists x \exists y(Rxy)$.