ANSWERS

BSc/MSci EXAMINATION

LOGIC

(CM328X Logic, 6CCM328A Logic, 5CCM328B Logic)

MAY-JUNE 2009

TIME ALLOWED: TWO HOURS

THIS PAPER CONSISTS OF TWO SECTIONS, SECTION A AND SECTION B.

SECTION A CONTRIBUTES HALF THE TOTAL MARKS FOR THE PAPER. ANSWER ALL QUESTIONS IN SECTION A.

QUESTIONS IN SECTION B CARRY EQUAL MARKS, BUT IF MORE THAN TWO ARE ATTEMPTED THEN ONLY THE BEST TWO WILL COUNT.

QUESTIONS SHOULD BE ANSWERED IN THE SPACES PROVIDED ON THE QUESTION PAPER.

YOU HAVE BEEN PROVIDED A BOOKLET FOR ROUGH WORK.
IT MUST BE ATTACHED TO THIS PAPER, BUT WILL NOT BE MARKED.

NO CALCULATORS ARE PERMITTED.

SECTION A

QUESTION 1 **(10 points)**

(a) Express each of the following in the notation of propositional logic. *Remark* : If in doubt leave blank, as each incorrect answer loses double the gain of a correct answer.

(i) if p then q, (ii) p if q, (iii) p only if q, (iv) only if p do we have q, (v) p iff q, (vi) p precisely if q.

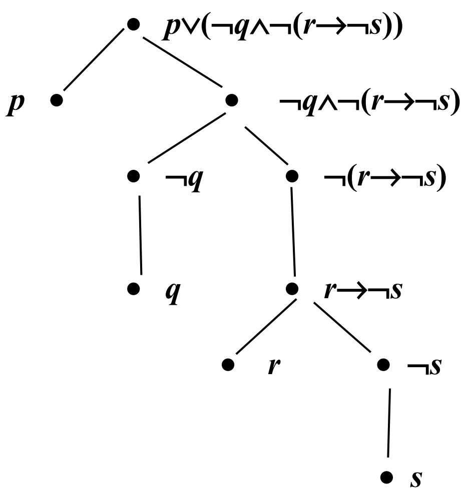
Answer (3 points worth ½ point each but 1 off for each incorrect answer):
(i) $p \to q$, (ii) $q \to p$, (iii) $p \to q$, (iv) $q \to p$, (v) $p \leftrightarrow q$, (vi) $p \leftrightarrow q$.

(b) What does it mean to say that a set of connectives of classical propositional logic is functionally complete? Explain briefly why the set $\{\neg,\wedge,\vee\}$ is functionally complete.

Answer (3 = 1+2 points): A set of connectives of classical propositional logic is functionally complete iff every truth-function (with a finite number of arguments) can be expressed by a formula built using only those connectives. The set $\{\neg,\wedge,\vee\}$ is functionally complete because if we take any truth function on *n* arguments we can express it as the disjunction of conjunctions of literals corresponding to the rows of the truth-table to which that function gives the value 1.

(c) Draw a syntactic decomposition tree for the propositional formula $p \vee (\neg q \wedge \neg(r \to \neg s))$.

Answer (3 points):



(d) Write the same formula $p \vee (\neg q \wedge \neg(r \to \neg s))$ in prefix (Polish) notation.

Answer (1 point): $\vee p \wedge \neg q \neg \to r \neg s$.

QUESTION 2 **(10 points)**

(a) Give an example to show that not every substitution instance of a contingent formula is contingent.

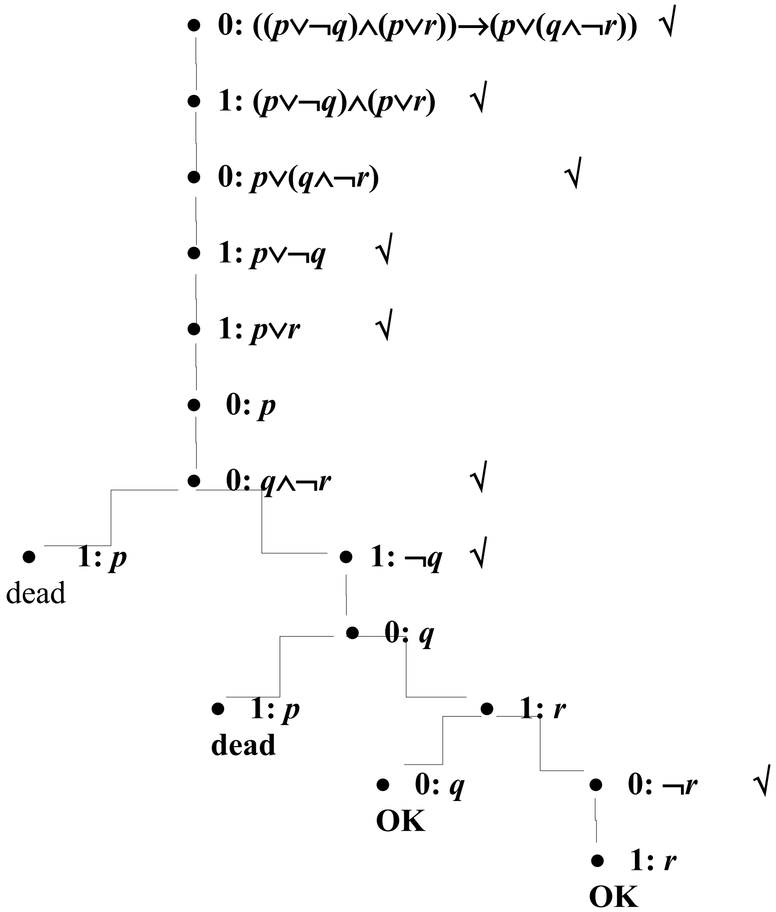
Answer (2 points): The simplest example is to take an elementary letter p, with $\sigma(p) = q\wedge\neg q$, say.

(b) Sketch a proof that every substitution instance of a tautology is a tautology.

Answer (3 points): Suppose that $\sigma(\alpha)$ is not a tautology. We want to show that α is not a tautology. By the supposition, there is a valuation v with $v(\sigma(\alpha)) = 0$. Let v' be the valuation that puts $v'(p) = v(\sigma(p))$ for every elementary letter p. Then it is straightforward to show by induction (on the recursive definition of the set of formulae) that $v'(\beta) = v(\sigma(\beta))$ for every formula β. In particular, $v'(\alpha) = v(\sigma(\alpha)) = 0$, so that α is not a tautology.

(c) Construct a semantic decomposition tree to determine whether or not the propositional formula $((p\vee\neg q)\wedge(p\vee r))\rightarrow(p\vee(q\wedge\neg r))$ is a tautology.

Answer (5 points): The semantic decomposition tree has an OK branch (in fact two), so the formula is not a tautology.



QUESTION 3 **(10 points)**

Let α be the formula $\exists y[\forall x(Rxy)\wedge Rzx]\vee\exists z[Py\rightarrow\forall z(Szx)]$.

(a) Identify the free occurrences of x,y,z in α.

Answer (2 points): As underlined $\exists y[\forall x(Rxy)\wedge R\underline{zx}]\vee\exists z[P\underline{y}\rightarrow\forall z(Sz\underline{x})]$.

(b) Let α be the same formula as in (a). Which of the following four substitutions are clean? $\alpha[y/x]$, $\alpha[f(w)/x]$, $\alpha[f(z)/x]$, $\alpha[x/y]$. *Remark* : If in doubt leave blank, as each incorrect answer loses double the gain of a correct answer.

Answer (2 points worth ½ point each but 1 off for each incorrect answer):

$\alpha[y/x]$	**No**
$\alpha[f(w)/x]$	**Yes**
$\alpha[f(z)/x]$	**No**
$\alpha[x/y]$	**Yes.**

(c) Define the concept of an x-variant of an interpretation for quantificational logic.

Answer: (2 points – either the short or the long formulation will do).
One interpretation is an *x*-variant of another iff they agree in their domain, the values assigned to all constants, function letters and predicate letters, and in the values assigned to all variables other than the variable *x*.
In other words, iff they disagree at most in the value assigned to the variable *x*.

(d) Formulate the x-variant reading of the existential quantifier.

Answer (2 points): $v(\exists x(\alpha)) = 1$ **where** $v = v_{D,\delta}$ **iff** $v_{D,\delta'}(\alpha) = 1$ **for some *x*-variant interpretation** (D,δ') **of** (D,δ).

(e) Formulate the substitutional reading of the existential quantifier, with attention to its proviso on the language used, and explaining briefly why that proviso is needed.

Answer (2 = 1+1 points):
$v(\exists x(\alpha)) = 1$ **iff** $v(\alpha[a/x]) = 1$ **for some constant symbol *a* of the language, with the proviso that the language has been expanded to contain at least one constant symbol for each element of the domain, assigned as its value.**
The proviso is needed in order to ensure that the existential quantifier ranges over the entire domain of discourse, and not just those elements that happen to have names.

QUESTION 4 **(10 points)**

Express the following statements in the language of quantificational logic with identity, in each case specifying a domain of discourse and a dictionary for all constants, function letters, and predicate letters employed.

(a) There is a set that is included in all sets, but no set includes all sets.

Answer (2 points):
Domain: sets. $x \subseteq y$: x is included in y.
Symbolization: $\exists x \forall y (x \subseteq y) \wedge \neg \exists y \forall x (x \subseteq y)$ or anything equivalent.

(b) The successors of distinct integers are always distinct.

Answer (2 points):
Domain: integers. $s(x)$: the successor of x; identity relation.
Symbolization: $\forall x \forall y [\neg(x \equiv y) \rightarrow \neg(s(x) \equiv s(y))]$ or $\forall x \forall y [(s(x) \equiv s(y)) \rightarrow (x \equiv y)]$.

(c) For any two distinct real numbers, exactly one is less than the other.

Answer (2 points):
Domain: real numbers, $x<y$: x is less than y; identity relation.
Symbolization: $\forall x \forall y [\neg(x \equiv y) \rightarrow ((x<y \vee y<x) \wedge \neg(x<y \wedge y<x))]$ or anything equivalent, e.g. using exclusive disjunction.

(d) Nobody loves everybody else.

Answer (2 points):
Domain: people. Lxy: x loves y; identity.
Symbolization: $\forall x \exists y (\neg(x \equiv y) \wedge \neg Lxy)$ or anything equivalent.

(e) Any candidate who can answer this question can answer all questions.
Answer (2 points):
Domain: any set the contains all people and all questions. Cx: x is a candidate; Qx: x is a question; Axy: x can answer y; a: this item.
Symbolization: $Qa \wedge \forall x [(Cx \wedge Axa) \rightarrow \forall y (Qy \rightarrow Axy)]$ or or anything equivalent. Approximate answers given partial credit for this one.

QUESTION 5 **(10 points)**

(a) Give (without calculations) an interpretation in a small finite domain that shows that $\exists x\exists y(Py\rightarrow Qx)$ does not logically imply $\exists y(Py)\rightarrow\exists x(Qx)$.

Answer (2 points): For example, domain: {1,2,3}, *R* = {(1,2), (2,3), (3,1)}.

(b) Show by natural deduction that $\forall x[\exists y(Py)\rightarrow\forall z(Rxz)]$ |- $\exists y(Py)\rightarrow\forall x(Rxx)$.

Answer (8 points):

n°	Formula	From	Rule	Depends on	Current goal
1	$\forall x[\exists y(Py)\rightarrow\forall z(Rxz)$]		**premise**	**1**	$\exists y(Py)\rightarrow\forall x(Rxx)$
2	$\exists y(Py)$		**supposition**	**2**	$\forall x(Rxx)$
3					Rxx
4	$\exists y(Py)\rightarrow\forall z(Rxz)$	**1**	**∀– proviso ok**	**1**	Rxx
5	$\forall z(Rxz)$	**2, 4**	**modus ponens**	**1, 2**	Rxx
6	Rxx	**5**	**∀– proviso ok**	**1, 2**	$\forall x(Rxx)$
7	$\forall x(Rxx)$	**1,2 |- 6**	**∀+ (proviso ok)**	**1, 2**	$\exists y(Py)\rightarrow\forall x(Qx)$
8	$\exists y(Py)\rightarrow\forall x(Qx)$	**1,2 |- 7**	**conditional proof**	**1 (2 dis)**	

SECTION B

QUESTION 6 **(25 points)**

(a) Identify a two-place truth-functional connective that is functionally complete when taken alone. Sketch a proof of its functional completeness (assuming the functional completeness of some familiar set of connectives).

Answer (8 points):
The following answer covers both of them, the candidates need only cover one.
There are two such connectives, *nand* (written $\uparrow$, true except when both components are true) and *neither…nor* (written $\downarrow$, false except when both components are false). We already know that $\{\neg,\wedge\}$ is functionally complete, as is also $\{\neg, \vee\}$, so it suffices to express either one of those. We may express $\neg\alpha$ as $\alpha\uparrow\alpha$, likewise as $\alpha\downarrow\alpha$. Using $\uparrow$, we can express $\wedge$ as $\neg(\alpha\uparrow\beta)$ i.e. as $(\alpha\uparrow\beta)\uparrow(\alpha\uparrow\beta)$, or $\vee$ as $(\neg\alpha\uparrow\neg\beta)$ i.e. as $(\alpha\uparrow\alpha)\uparrow(\beta\uparrow\beta)$. Likewise we can express $\vee$ as $\neg(\alpha\downarrow\beta)$ i.e. as $(\alpha\downarrow\beta)\downarrow(\alpha\downarrow\beta)$, or $\wedge$ as $(\neg\alpha\downarrow\neg\beta)$ i.e. as $(\alpha\downarrow\alpha)\downarrow(\beta\downarrow\beta)$.

(b) Sketch a proof that the set $\{\vee,\wedge,\rightarrow,\leftrightarrow\}$ of propositional connectives is not functionally complete.

Answer (8 points): We claim that these connectives cannot express negation (nor, indeed, any other truth-function that has value 1 in the top row of its truth-table). It suffices to show that no formula generated by these connectives is tautologically equivalent to $\neg p$. Consider the valuation v that gives every elementary letter the value 1. By induction, $v(\alpha) = 1$ for every such formula, whereas $v(\neg p) = 0$.

(c) Use successive transformations to rewrite $\neg[r\wedge(q\rightarrow\neg p)]$ in disjunctive normal form.

Answer (3 points): $\neg[r\wedge(q\rightarrow\neg p)]$ -||- $\neg r\vee\neg(q\rightarrow\neg p)$ -||- $\neg r\vee(q\wedge\neg\neg p)$ -||- $\neg r\vee(q\wedge p)$.

(d) Without writing out a proof, identify any redundant letters in the formula $(p\vee\neg r)\vee(\neg q\wedge p)$ and express it in least letter-set form.

Answer (3 points): The redundant letter is q, and the formula is equivalent to $p\vee\neg r$.

(e) Without writing out a proof, express the set $A = \{(p\wedge(q\rightarrow s))\vee r, \neg r\}$ of formulae in most modular (alias finest splitting) form.

Answer (3 points): Most modular form: $\{p, (q\rightarrow s), \neg r\}$.

QUESTION 7 **(25 points)**

(a) Write out the four quantifier interchange equivalences. To what propositional equivalences do they correspond?

Answer (5 = 4+1 points):

LHS -‖- RHS	
$\neg\forall x(\alpha)$	$\exists x(\neg\alpha)$
$\neg\exists x(\alpha)$	$\forall x(\neg\alpha)$
$\forall x(\alpha)$	$\neg\exists x(\neg\alpha)$
$\exists x(\alpha)$	$\neg\forall x(\neg\alpha)$

They correspond to the four de Morgan equivalences in propositional logic.

(b) State the rule $\forall-$, with careful attention to its proviso.

Answer (4 points): $\forall x(\alpha)$ |- $\alpha[t/x]$, provided the substitution $\alpha[t/x]$ is clean.

(c) Which of the following are instances of the rule $\forall-$? Answer 'yes' or 'no' to the right of each. *Remark* : If in doubt leave blank, as each incorrect answer loses double the gain of a correct answer

$\forall x\exists y(Px\rightarrow Qxy)$ |- $\exists y(Px\rightarrow Qzy)$
same formula |- $\exists y(Py\rightarrow Qyy)$
same formula |- $\exists y(Pz\rightarrow Qzy)$
same formula |- $\exists y(Pa\rightarrow Qay)$

Answer (4 points):
No (not a uniform substitution)
No (not clean)
Yes (clean)
Yes (substitution of a constant, so clean)

(d) State the indirect rule $\forall+$, with careful attention to its proviso.

Answer (3 points): Whenever $\alpha_1,\ldots,\alpha_n$ |- α then $\alpha_1,\ldots,\alpha_n$ |- $\forall x(\alpha)$, provided the variable x has no free occurrences in any of $\alpha_1,\ldots,\alpha_n$.

(e) Which of the following implications are justified by the rule $\forall+$, applied to the implication $\forall x\exists y(Rxyz)$ |- $\exists y(Rwyz)$? Answer 'yes' or 'no' to the right of each. *Remark* : If in doubt leave blank, as each incorrect answer loses double the gain of a correct answer.

$\forall x \exists y(Rxyz)$ |- $\forall w \exists y(Rwyz)$
$\forall x \exists y(Rxyz)$ |- $\forall z \exists y(Rwyz)$
$\forall x \exists y(Rxyz)$ |- $\forall y \exists y(Rwyz)$

Answer (3 points):
Yes (since *w* not free on LHS)
No (*z* occurs free on LHS)
Yes (although vacuous).

(f) State the rule of replacement for identity in quantificational logic, with careful attention to its proviso.

Answer (3 points): α, $t \equiv t'$ |- $\alpha[t'//t]$, provided the replacement is clean.

(g) Let α be the formula $\forall y(R(y,f(x,y)))$ and let t be the variable x. Write out $\alpha[t'//t]$ for the following three choices of term t', and in each case state whether the rule of replacement authorizes the implication α, $t \equiv t'$ |- $\alpha[t'//t]$. *Remark* : If in doubt leave blank, as each incorrect answer loses double the gain of a correct answer.

(i) Let t' be the term $g(z)$
(ii) Let t' be the constant a
(iii) Let t' be the variable y

Answer (3 points):

$\forall y(R(y,f(g(z),y)))$	**yes**
$\forall y(R(y,f(a,y)))$	**yes**
$\forall y(R(y,f(y,y)))$	**no (not clean)**

QUESTION 8 **(25 points)**

(a) Use natural deduction to show that $\forall x \forall y(Rxy)$ |- $\forall x \forall y(Ryx)$. *Remark*: Be careful with your applications of $\forall-$.

Answer (12 points)

n°	Formula	From	Rule	Depends on	Current goal
1	$\forall x \forall y(Rxy)$		premise	1	$\forall x \forall y(Ryx)$
2					$\forall y(Ryx)$
3					Ryx
4	$\forall y(Rzy)$	1	$\forall-$ proviso ok	1	Ryx
5	Rzx	4	$\forall-$ proviso ok	1	Ryx
6	$\forall z(Rzx)$	1 |- 5	$\forall+$ proviso ok	1	Ryx
7	Ryx	6	$\forall-$ proviso ok	1	$\forall y(Ryx)$
8	$\forall y(Ryx)$	1 |- 7	$\forall+$ proviso ok	1	$\forall x \forall y(Ryx)$
9	$\forall x \forall y(Ryx)$	1 |- 8	$\forall+$ proviso ok	1	

(b) Use natural deduction centring on proof by contradiction to show that $\forall x\forall y\forall z(Rxy\rightarrow\neg Rxz)$ |- $\neg\exists x\forall y(Rxy)$.

Answer (13 points)

n°	Formula	From	Rule	Depends on	Current goal
1	$\forall x\forall y\forall z(Rxy\rightarrow\neg Rxz)$		**premise**	**1**	$\neg\exists x\forall y(Rxy)$
2	$\exists x\forall y(Rxy)$		**supposition**	**2**	$\perp$
3	$\forall y(Rxy)$		**supposition**	**3**	$\perp$
4	Rxx	**3**	**$\forall-$** **proviso ok**	**3**	$\perp$
5	$Rxx\rightarrow\neg Rxx$	**1**	**$\forall-$ (thrice)** **proviso ok**	**1**	$\perp$
6	$\neg Rxx$	**4, 5**	**mp**	**1, 3**	$\perp$
7	$Rxx\wedge\neg Rxx$	**4,6**	**$\wedge$+**	**1, 3**	$\perp$
8	$Rxx\wedge\neg Rxx$	**1,3 |- $\perp$**	**$\exists-$ ($\perp$)** **proviso ok**	**1, 2** **(3 dis)**	$\neg\exists x\forall y(Rxy)$
9	$\neg\exists x\forall y(Rxy)$	**1,2 |- $\perp$**	**raa**	**1** **(2 dis)**	

QUESTION 9 **(25 points)**

(a) State the compactness theorem for quantificational logic. Which half of it is trivial, and why?

Answer (5 points): A set *A* of formulae is consistent iff every finite subset of *A* is consistent. The half LHS ⇒ RHS is trivial, for if there is a model satisfying the whole set, it satisfies all of its subsets, thus in particular each of the finite subsets.

(b) Explain in rough terms what it means to say that a relation between formulae is decidable. Why is tautological entailment decidable? Is entailment in the context of quantificational logic decidable (yes or no)?

Answer (5 points): A relation between formulae is decidable iff there is an algorithm which, for any pair (α,β) of formulae, determines in a finite time whether or not the pair is in the relation. Tautological consequence is decidable because such an algorithm is provided by the method of truth-tables (another is the method of semantic decomposition trees). However, entailment in the context of quantificational logic is not decidable.

(c) What is prenex normal form? What is the prenex normal form theorem?

Answer (5 points): A formula of quantificational logic is in prenex normal form when it is of the form $Q_1x_1 \ldots Q_nx_n(\alpha)$ where the Q_i are quantifiers, the x_i are variables, and there are no quantifiers in α. The prenex normal theorem states that every formula of quantificational logic is equivalent to one in prenex normal form.

(d) What does it mean to say that our system of natural deduction for quantificational logic is (i) sound and (ii) complete with respect to logical implication?

Answer (5 points):
(i) Whenever α is derivable from *A* by means of the rules of the system a set *A* of formulae, then *A* logically implies α.
(ii) The converse: whenever a set *A* of formulae logically implies a formula α, then α is derivable from *A* by means of the rules of the system.

(e) What is the Löwenheim-Skolem theorem, and why is it sometimes regarded as paradoxical?

Answer (5 points): If a set *A* of formulae of quantificational logic is consistent then there is an interpretation δ over a countable domain *D* such that $v_{D,\delta}(\forall x(\alpha)) = 1$ for all formula α in *A*. This is sometimes regarded as paradoxical because there are mathematical theories such as set theory formalized in the language of quantificational logic, in which we can prove the existence of non-countable sets; but by the Löwenheim-Skolem theorem, if such theories are consistent they have countable models.