BSc/MSci EXAMINATION

LOGIC

(CM328X Logic, 6CCM328A Logic, 5CCM328B Logic)

MAY-JUNE 2009

TIME ALLOWED: TWO HOURS

THIS PAPER CONSISTS OF TWO SECTIONS, SECTION A AND SECTION B.

SECTION A CONTRIBUTES HALF THE TOTAL MARKS FOR THE PAPER. ANSWER ALL QUESTIONS IN SECTION A.

QUESTIONS IN SECTION B CARRY EQUAL MARKS, BUT IF MORE THAN TWO ARE ATTEMPTED THEN ONLY THE BEST TWO WILL COUNT.

QUESTIONS SHOULD BE ANSWERED IN THE SPACES PROVIDED ON THE QUESTION PAPER.

YOU HAVE BEEN PROVIDED A BOOKLET FOR ROUGH WORK.
IT MUST BE ATTACHED TO THIS PAPER, BUT WILL NOT BE MARKED.

NO CALCULATORS ARE PERMITTED.

SECTION A

QUESTION 1 **(10 points)**

(a) Express each of the following in the notation of propositional logic. *Remark* : If in doubt leave blank, as each incorrect answer loses double the gain of a correct answer.

(i) if p then q, (ii) p if q, (iii) p only if q, (iv) only if p do we have q, (v) p iff q, (vi) p precisely if q.

(b) What does it mean to say that a set of connectives of classical propositional logic is functionally complete? Explain briefly why the set $\{\neg,\wedge,\vee\}$ is functionally complete.

(c) Draw a syntactic decomposition tree for the propositional formula $p\vee(\neg q\wedge\neg(r\rightarrow\neg s))$.

(d) Write the same formula $p\vee(\neg q\wedge\neg(r\rightarrow\neg s))$ in prefix (Polish) notation.

QUESTION 2 **(10 points)**

(a) Give an example to show that not every substitution instance of a contingent formula is contingent.

(b) Sketch a proof that every substitution instance of a tautology is a tautology.

(c) Construct a semantic decomposition tree to determine whether or not the propositional formula $((p\vee\neg q)\wedge(p\vee r))\rightarrow(p\vee(q\wedge\neg r))$ is a tautology.

QUESTION 3 **(10 points)**

Let α be the formula $\exists y[\forall x(Rxy)\wedge Rzx]\vee\exists z[Py\rightarrow\forall z(Szx)]$.

(a) Identify the free occurrences of x,y,z in α.

(b) Let α be the same formula as in (a). Which of the following four substitutions are clean? $\alpha[y/x]$, $\alpha[f(w)/x]$, $\alpha[f(z)/x]$, $\alpha[x/y]$. *Remark* : If in doubt leave blank, as each incorrect answer loses double the gain of a correct answer.

(c) Define the concept of an x-variant of an interpretation for quantificational logic.

(d) Formulate the x-variant reading of the existential quantifier.

(e) Formulate the substitutional reading of the existential quantifier, with attention to its proviso on the language used, and explaining briefly why that proviso is needed.

QUESTION 4 **(10 points)**

Express the following statements in the language of quantificational logic with identity, in each case specifying a domain of discourse and a dictionary for all constants, function letters, and predicate letters employed.

(a) There is a set that is included in all sets, but no set includes all sets.

(b) The successors of distinct integers are always distinct.

(c) For any two distinct real numbers, exactly one is less than the other.

(d) Nobody loves everybody else.

(e) Any candidate who can answer this question can answer all questions.

QUESTION 5 **(10 points)**

(a) Give (without calculations) an interpretation in a small finite domain that shows that $A = \{\forall x \exists y(Rxy), \forall x \exists y(Ryx)\}$ does not logically imply $\exists x \exists y(Rxy \wedge Ryx)$.

(b) Show by natural deduction that $\forall x(\alpha \rightarrow \beta) \mid\!- \exists x(\alpha) \rightarrow \exists x(\beta)$ for arbitrary formulae α, β.

SECTION B

QUESTION 6 **(25 points)**

(a) Identify a two-place truth-functional connective that is functionally complete when taken alone. Sketch a proof of its functional completeness (assuming the functional completeness of some familiar set of connectives).

(b) Sketch a proof that the set $\{\vee,\wedge,\rightarrow,\leftrightarrow\}$ of propositional connectives is not functionally complete.

(c) Use successive transformations to rewrite $\neg[r\wedge(q\rightarrow\neg p)]$ in disjunctive normal form.

(d) Without writing out a proof, identify any redundant letters in the formula $(p\vee\neg r)\vee(\neg q\wedge p)$ and express it in least letter-set form.

(e) Without writing out a proof, express the set $A = \{(p\wedge(q\rightarrow s))\vee r, \neg r\}$ of formulae in most modular (alias finest splitting) form.

QUESTION 7 **(25 points)**

(a) Write out the four quantifier interchange equivalences. To what propositional equivalences do they correspond?

(b) State the rule $\forall-$, with careful attention to its proviso.

(c) Which of the following are instances of the rule $\forall-$? Answer 'yes' or 'no' to the right of each. *Remark* : If in doubt leave blank, as each incorrect answer loses double the gain of a correct answer

$\forall x\exists y(Px\rightarrow Qxy)$ |- $\exists y(Px\rightarrow Qzy)$
same formula |- $\exists y(Py\rightarrow Qyy)$
same formula |- $\exists y(Pz\rightarrow Qzy)$
same formula |- $\exists y(Pa\rightarrow Qay)$

(d) State the indirect rule $\forall+$, with careful attention to its proviso.

(e) Which of the following implications are justified by the rule $\forall+$, applied to the implication $\forall x\exists y(Rxyz)$ |- $\exists y(Rwyz)$? Answer 'yes' or 'no' to the right of each. *Remark* : If in doubt leave blank, as each incorrect answer loses double the gain of a correct answer.

$\forall x\exists y(Rxyz)$ |- $\forall w\exists y(Rwyz)$
$\forall x\exists y(Rxyz)$ |- $\forall z\exists y(Rwyz)$
$\forall x\exists y(Rxyz)$ |- $\forall y\exists y(Rwyz)$

(f) State the rule of replacement for identity in quantificational logic, with careful attention to its proviso.

(g) Let α be the formula $\forall y(R(y,f(x,y)))$ and let t be the variable x. Write out $\alpha[t'//t]$ for the following three choices of term t', and in each case state whether the rule of replacement authorizes the implication $\alpha,\ t \equiv t'\ \vdash\ \alpha[t'//t]$. *Remark* : If in doubt leave blank, as each incorrect answer loses double the gain of a correct answer.

(i) Let t' be the term $g(z)$
(ii) Let t' be the constant a
(iii) Let t' be the variable y

QUESTION 8 **(25 points)**

(a) Use natural deduction to show that $\forall x \forall y(Rxy) \vdash \forall x \forall y(Ryx)$. *Remark*: Be careful with your applications of $\forall-$.

(b) Use natural deduction centring on proof by contradiction to show that $\forall x \forall y \forall z(Rxy \rightarrow \neg Rxz)$ |- $\neg \exists x \forall y(Rxy)$.

QUESTION 9 **(25 points)**

(a) State the compactness theorem for quantificational logic. Which half of it is trivial, and why?

(b) Explain in rough terms what it means to say that a relation between formulae is decidable. Why is tautological entailment decidable? Is entailment in the context of quantificational logic decidable (yes or no)?

(c) What is prenex normal form? What is the prenex normal form theorem?

(d) What does it mean to say that our system of natural deduction for quantificational logic is (i) sound and (ii) complete with respect to logical implication?

(e) What is the Löwenheim-Skolem theorem, and why is it sometimes regarded as paradoxical?