The Handbook of Mathematics, Physics and Astronomy Data is provided

KEELE UNIVERSITY

EXAMINATIONS, 2012/13

Level III

Thursday 2nd May 2013, 09.30-11.30

PHYSICS/ASTROPHYSICS

PHY-30028

PHYSICS OF GALAXIES

Candidates should attempt to answer THREE questions.

A sheet of useful formulae can be found on the last page.

NOT TO BE REMOVED FROM THE EXAMINATION HALL

1. (a) Consider a system of main-sequence stars that have individual masses $M_\star$ between the limits $M_\ell \le M_\star \le M_u$, distributed according to $dN/dM_\star = A\, M_\star^{-\beta}$ with A and β constants.

   i. Assuming the approximate mass–luminosity relation $L_\star \approx M_\star^3$ (in Solar units) for individual main-sequence stars, show that the total mass-to-light ratio of the stellar system is
   $$\frac{M_{\rm tot}}{L_{\rm tot}} \approx \frac{4-\beta}{2-\beta}\,\frac{M_u^{2-\beta} - M_\ell^{2-\beta}}{M_u^{4-\beta} - M_\ell^{4-\beta}} \ . \qquad [20]$$

   ii. Estimate the mass-to-light ratio of an old stellar system with $\beta = 1.2$, $M_\ell = 0.08\ M_\odot$ and $M_u = 0.9\ M_\odot$. [5]

   (b) Write down the general equation for the acceleration $\ddot{\boldsymbol{x}}$ in a gravitational potential $\Phi(\boldsymbol{x})$. Use this to show that the speed on a circular orbit of radius R in a plane at fixed z in an axisymmetric galaxy is given by $V_c^2(R,z) = R\left(\partial\Phi/\partial R\right)$. [20]

   (c) A power-law model for the potential of the Milky Way is
   $$\Phi(R,z) \;=\; -\,C\left(R^2 + z^2/q^2\right)^{-n} \ ,$$
   where C, q and n are positive constants.

   i. Show that the circular speed in the midplane ($z = 0$) of the Galaxy, in this model, is $V_c(R,0) = \sqrt{2nC}\ R^{-n}$. [15]

   ii. At radius R in an axisymmetric disc, the Oort constants are
   $$A \equiv -\frac{1}{2}\left[\frac{dV_c}{dR} - \frac{V_c}{R}\right] \qquad \text{and} \qquad B \equiv -\frac{1}{2}\left[\frac{dV_c}{dR} + \frac{V_c}{R}\right] \ .$$
   Near the Sun, observations give $A = 14.8$ km s^{-1} kpc^{-1} and $B = -12.4$ km s^{-1} kpc^{-1}. Thus, calculate V_c at the position of the Sun, and infer the value of the exponent n in a power-law model for the Milky Way potential. [20]

   iii. Within this power-law model, calculate the circular speed at $R = 50$ kpc in the plane of the Milky Way. Given that the total luminosity of the Galaxy is $L_{\rm tot} \approx 2 \times 10^{10}\ L_\odot$, estimate the mass of dark matter contained within 50 kpc. [20]

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2. The Plummer model for a self-gravitating, spherical mass distribution has the potential and density

$$\Phi(r) \;=\; -\frac{G\mathcal{M}}{(r^2+a^2)^{1/2}} \qquad \text{and} \qquad \rho(r) \;=\; \frac{3\mathcal{M}}{4\pi}\,\frac{a^2}{(r^2+a^2)^{5/2}}\;,$$

where $\mathcal{M}$ and a are constants.

(a) Write down the equation used to obtain $\rho(r)$ from $\Phi(r)$ (no calculation is required). [5]

(b) Show that the mass contained within radius r in a Plummer sphere is

$$M(r) \;=\; \mathcal{M}\,\frac{r^3}{(r^2+a^2)^{3/2}}\;. \qquad [15]$$

(c) Show that the half-mass radius of a Plummer sphere is

$$r_h \;\simeq\; 1.3048\,a\;. \qquad [15]$$

(d) The total gravitational potential energy of a spherical mass distribution is given in general by

$$W \;=\; \frac{1}{2}\int_0^\infty \Phi(r)\,\rho(r)\,4\pi r^2\,dr\;.$$

i. Explain this equation in physical terms. [10]

ii. Show that, for the Plummer model,

$$W \;\simeq\; -\,0.3843\,\frac{G\mathcal{M}^2}{r_h}\;. \qquad [25]$$

(e) The globular star cluster ω Centauri has a V-band luminosity $L_V = 1.3\times10^6\ L_\odot$; a half-mass radius $r_h = 9.7$ pc; and an average one-dimensional stellar velocity dispersion $\sigma = 12$ km s^{-1}. The dwarf galaxy Sculptor has $L_V = 2.3\times10^6\ L_\odot$ and $\sigma = 9.9$ km s^{-1}, similar to ω Centauri, but $r_h = 380$ pc.

Assuming that the stars in both ω Centauri and Sculptor are old, and that Plummer models describe the mass distributions of both systems, use the virial theorem to estimate the mass of dark matter in each. [30]

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3. (a) What is meant by the Eddington limit? Under what conditions might the Eddington luminosity be exceeded? [15]

(b) An accreting black hole of mass M has a luminosity L.

i. Write down equations for the forces acting on a proton–electron plasma at a radius of r from the black hole. [10]

ii. Thus, show that the luminosity ($L_{\rm Edd}$) of an accreting black hole of mass M at the Eddington limit is given by

$$L_{\rm Edd} = \frac{4\pi G c M m_{\rm p}}{\sigma_{\rm T}} \ ,$$

where $\sigma_{\rm T}$ is the Thomson cross section. [10]

(c) Consider a particle of mass m in an accretion disc at a radius r from a black hole of mass M.

i. Write down an expression for the gravitational potential energy of the particle. [5]

ii. Use the virial theorem to show that the total luminosity L emitted by a disc around the black hole is

$$L = \frac{GM\dot{M}}{2r} \ ,$$

where $\dot{M}$ is the accretion rate onto the black hole. [15]

(d) The black hole in the quasar 3C 273 has a mass of $M = 10^9\ {\rm M}_\odot$ and accretes at the Eddington limit.

i. Estimate the luminosity of 3C 273. [5]

ii. Calculate the mass accretion rate of 3C 273 (in Solar units) if the accretion disc extends down to the innermost stable circular orbit (ISCO) of a Schwarzschild black hole. [20]

iii. What is the corresponding efficiency factor of converting rest mass into energy in this case? [10]

iv. How might the efficiency differ for a Kerr black hole? [10]

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4. (a) A virialised, three-dimensional distribution of clouds in the Br Line Region (BLR) of an AGN has a line-of-sight velocity dispersion σ. If r is the radial distance of the clouds from the black hole in the AGN, show that the mass of the black hole is

$$M_{\mathrm{BH}} = \frac{3\sigma^2 r}{G} \quad .$$ [15]

(b) The broad Hβ line of the Seyfert galaxy, NGC 5548, is measured in a reverberation observation to determine its black hole mass.

i. The Hβ line in NGC 5548 has an emitted wavelength of 486.1 nm and has a Full Width Half Maximum of 8.10 nm. Calculate the velocity dispersion of the BLR clouds. [15]

ii. The variations in the Hβ line are delayed with respect to the AGN continuum. Explain why this delay occurs. [10]

iii. If the time delay between the continuum and Hβ emission is 10 days, estimate the radius of the BLR and hence calculate the mass of the black hole in NGC 5548. [10]

(c) Suppose that the *stars* around a black hole of mass M_{BH} have the density distribution $\rho_\star(r) = K\, r^{-2}$, with K a constant.

i. Solve the spherical Jeans equation to show that the one–dimensional velocity dispersion of the stars at radius r from the black hole is given by

$$\sigma_\star^2(r) \;=\; 2\pi\, G\, K + \frac{1}{3}\,\frac{G\, M_{\mathrm{BH}}}{r} \quad .$$ [25]

ii. Taking $K = 1.5 \times 10^5\ M_\odot\ \mathrm{pc}^{-1}$ for stars near the centre of NGC 5548, and using the value of M_{BH} from part (b), calculate the radius r_{eq} at which the contribution of the black hole to $\sigma_\star$ equals the contribution of the stars themselves. Evaluate $\sigma_\star$ at some appropriate radii in order to make a sketch of $\sigma_\star$ versus r with scales on both axes. [25]

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5. The hot diffuse plasma in a galaxy cluster can emit X-ray radiat
via Bremsstrahlung, whereby the total luminosity L is given by

$$L = 1.4 \times 10^{-40} Z^2 T^{0.5} \int n_e^2 \, dV \quad ,$$

where Z is the average atomic number of the gas, T is its temperature and n_e is the electron density. The integral $\int n_e^2 \, dV$ is referred to as the *emission measure* of the plasma.

(a) i. State how Bremsstrahlung radiation is emitted. Why is the emission measure proportional to n_e^2? [15]

ii. Explain how the luminosity of the X-ray Bremsstrahlung radiation might be affected by cluster mass. [15]

(b) The electron density of a cluster varies with radius r as a power-law of the form $n_e(r) = n_c(r/r_c)^{-\alpha}$, where n_c is the core density and r_c is the core radius of the cluster.

Show that the emission measure of the cluster is equal to

$$\int_0^R n_e^2 \, dV = \frac{4\pi n_c^2 r_c^{2\alpha}}{(3-2\alpha)} R^{(3-2\alpha)} \quad . \qquad [15]$$

(c) The Coma cluster has a core radius of $r_c = 200\,\mathrm{kpc}$, $kT = 8\,\mathrm{keV}$ and an X-ray luminosity of $L = 10^{38}\,\mathrm{W}$, while $\alpha = 1.3$.

i. Evaluate the emission measure of the cluster out to a radius of $R = 2\,\mathrm{Mpc}$ and hence determine the core electron density (assume the gas is composed of hydrogen). [20]

ii. Calculate the total mass of X-ray emitting gas in the cluster inside a radius of 2 Mpc. [25]

(d) If the total B-band luminosity of the cluster is $L_B = 5 \times 10^{12}\, L_\odot$, estimate what fraction of the total baryonic mass consists of hot X-ray emitting gas. [10]

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<u>Some useful formulae</u>

In spherical polar coordinates,

$$\ddot{\boldsymbol{x}} \;=\; \widehat{\boldsymbol{r}}\left(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2\right) \;+\; \widehat{\boldsymbol{\theta}}\left(r\ddot{\theta} + 2\,\dot{r}\dot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2\right) \;+\; \widehat{\boldsymbol{\phi}}\left(r\sin\theta\,\ddot{\phi} + 2\sin\phi\,\dot{r}\dot{\phi} + 2\,r\cos\theta\,\dot{\theta}\dot{\phi}\right)$$

$$\boldsymbol{\nabla}\Phi \;=\; \widehat{\boldsymbol{r}}\,\frac{\partial\Phi}{\partial r} \;+\; \widehat{\boldsymbol{\theta}}\,\frac{1}{r}\frac{\partial\Phi}{\partial\theta} \;+\; \widehat{\boldsymbol{\phi}}\,\frac{1}{r\,\sin\theta}\frac{\partial\Phi}{\partial\phi}$$

$$\nabla^2\Phi \;=\; \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) \;+\; \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\Phi}{\partial\theta}\right) \;+\; \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

In cylindrical polar coordinates,

$$\ddot{\boldsymbol{x}} \;=\; \widehat{\boldsymbol{R}}\left(\ddot{R} - R\dot{\theta}^2\right) \;+\; \widehat{\boldsymbol{\theta}}\left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right) \;+\; \widehat{\boldsymbol{z}}\,\ddot{z}$$

$$\boldsymbol{\nabla}\Phi \;=\; \widehat{\boldsymbol{R}}\,\frac{\partial\Phi}{\partial R} \;+\; \widehat{\boldsymbol{\theta}}\,\frac{1}{R}\frac{\partial\Phi}{\partial\theta} \;+\; \widehat{\boldsymbol{z}}\,\frac{\partial\Phi}{\partial z}$$

$$\nabla^2\Phi \;=\; \frac{1}{R}\frac{\partial}{\partial R}\left(R\,\frac{\partial\Phi}{\partial R}\right) \;+\; \frac{1}{R^2}\frac{\partial^2\Phi}{\partial\theta^2} \;+\; \frac{\partial^2\Phi}{\partial z^2}$$

Spherical Jeans equation: $$\frac{d}{dr}\left(\rho\sigma^2\right) \;=\; -\rho(r)\,\frac{GM(r)}{r^2}$$

Standard integrals: $$\int_0^\infty \frac{x^2}{(1+x^2)^3}\,dx \;=\; \frac{\pi}{16}$$

$$\int \frac{x^2}{(1+x^2)^{5/2}}\,dx \;=\; \frac{1}{3}\,\frac{x^3}{(1+x^2)^{3/2}}$$