The Handbook of Mathematics, Physics and Astronomy Data is provided

KEELE UNIVERSITY

EXAMINATIONS, 2012/13

Level III

Wednesday 9th January 2013, 09.30-11.30

PHYSICS

PHY-30012

ELECTROMAGNETISM

Candidates should attempt to answer THREE questions.

A sheet of useful formulae and vector identities can be found on the last page.

NOT TO BE REMOVED FROM THE EXAMINATION HALL

1. (a) Starting from Faraday's law in differential form, show that, the case of a vacuum, a wave equation of the form

$$\nabla^2 \mathbf{E} = \frac{1}{v^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

can be obtained, and derive an expression for the wave speed, v. [20]

(b) An electric field is given by

$$\mathbf{E} = E_0 \cos(kz - \omega t)\hat{\mathbf{i}} \; .$$

Show that this E-field obey's Gauss's law in vacuum and is a solution to the equation in part (a). [30]

(c) Derive an expression for the associated time-dependent magnetic field by using Maxwell's equations in differential form. [20]

(d) If the z-axis is aligned with the Earth's gravitational field and $E_0 = 10^6\,\mathrm{V\,m^{-1}}$ then, stating any assumptions you make, estimate the size of the largest cubic, black particle, of density $3000\ \mathrm{kg\,m^{-3}}$, that could be supported against gravity by the radiation pressure exerted by these fields in a laboratory vacuum. [30]

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2. (a) Explain what is meant by the oscillating *electric dipole* *proximation* and discuss, with reasons, the circumstances under which the approximation is valid. [15]

(b) How can the electric *dipole* approximation be used in estimating the electromagnetic fields from an oscillating electric *quadrupole*? [10]

(c) The magnetic field, $\mathbf{B}(\mathbf{r}, t)$ due to an oscillating electric quadrupole is given in spherical polar coordinates by

$$\mathbf{B} = \frac{10^{-3}}{r} \sin\theta \cos\theta \, \cos[\omega(t - r/c)] \, \hat{\boldsymbol{\phi}} \; \mathrm{T} \, ,$$

where r is the distance from the quadrupole and the other symbols have their usual meanings.

i. Write down, with justification, an expression for the associated electric field. [20]

ii. Derive an expression for the *time-averaged* Poynting vector. [15]

iii. Estimate what fraction of the total electromagnetic power emitted by the quadrupole would be absorbed by a black circular disc of radius 4 m that is positioned 10 m along the z-axis with its flat face in the x, y plane. [40]

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3. (a) Using Faraday's law in integral form, show that the component of the electric field which is tangential to the interface between two different media is continuous. [20]

(b) A square, parallel-plate capacitor is of side L, and its plates are separated by d (where $d \ll L$) and hold a charge $\pm Q$.

Sketch the capacitor, with electric field lines, and use suitable Gaussian surfaces to show that the electric field between the plates has a magnitude $E = Q/(\epsilon_0 L^2)$, while outside the plates it is zero. [20]

(c) A LIH dielectric material with width just less than d and relative permittivity ϵ_r is then *partially* inserted between the plates to a distance b, so that an area bL lies inside the plates. If the *total* charge on the plates is maintained at $\pm Q$ then:

i. Make another sketch of the capacitor showing both electric field lines and field lines that represent the displacement field $\mathbf{D}$. Indicate the relative distribution of charge on the capacitor plates. [15]

ii. Use Gauss's law in media to estimate the strength of the E- and D-fields that you have drawn. [20]

iii. The energy density in the fields is given by $\mathbf{E} \cdot \mathbf{D}/2$. Hence or otherwise, show that the electrical energy stored by the capacitor is

$$U = \frac{Q^2}{2\epsilon_0}\left(\frac{d}{L^2 + (\epsilon_r - 1)bL}\right)$$ [15]

iv. If the susceptibility of the dielectric is $\chi_e = 9$, what is the ratio of charge surface density on that part of the capacitor plate immediately above the dielectric to that part still above a vacuum? [10]

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4. (a) Explain, using a "thought experiment" why electric and m... netic fields cannot be invariant in different, uniformly moving reference frames. [20]

(b) The electric and magnetic fields observed in frame S' moving with uniform velocity $\mathbf{v}$ with respect to another frame S are given by

$$\begin{aligned} \mathbf{E}' &= \mathbf{E}_{\|} + \gamma\left[\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}\right] , \\ \mathbf{B}' &= \mathbf{B}_{\|} + \gamma\left[\mathbf{B}_{\perp} - \frac{1}{c^2}(\mathbf{v} \times \mathbf{E})\right] , \end{aligned}$$

where the $\|$ and $\perp$ subscripts refer to components that are either parallel or perpendicular to $\mathbf{v}$ and $\gamma = (1 - v^2/c^2)^{-1/2}$.

Assuming that $\mathbf{v} = 2.95 \times 10^8\, \hat{\mathbf{i}}\ \mathrm{m\,s^{-1}}$, then for sinusoidal electromagnetic waves in vacuum travelling in the *negative* x direction:

i. Write down an expression for the E-field in frame S and calculate $\mathbf{E}'$ in terms of transformed space and time coordinates. (The Lorentz transforms can be found in the Maths handbook). [40]

ii. Show that the wave speed is the same in S and S'. [10]

iii. Assuming that the relationships between the E- and B-fields of an electromagnetic wave are frame-invariant, then calculate the ratio of the time-averaged Poynting vectors in the two reference frames. [20]

iv. Give an example of why this phenomenon is important in interpreting the radiation seen from an astrophysical source. [10]

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5. In an electrically neutral, non-magnetic, conducting LIH medium

$$\nabla^2 \mathbf{E} = \mu_0 \, \sigma \, \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \, \epsilon_r \, \epsilon_0 \, \frac{\partial^2 \mathbf{E}}{\partial t^2} \, ,$$

where σ is the conductivity and the other symbols have their usual meanings.

(a) Show that $\mathbf{E} = E_o \exp(-\beta x) \exp[i(\alpha x - \omega t)] \, \hat{\mathbf{j}}$ is a solution to this differential equation and in a very good conductor, where $\sigma \gg \epsilon_r \epsilon_0 \omega$, that $\alpha = \beta = (\mu_0 \sigma \omega / 2)^{1/2}$. [30]

(b) Explain what is meant by the "skin depth" and calculate its value for a wave with frequency 200 MHz in silver, where $\sigma = 6.17 \times 10^7 \, \mathrm{S \, m^{-1}}$. [10]

(c) A travelling wave in vacuum, of electric field amplitude $E_i = 100 \, \mathrm{V \, m^{-1}}$ and frequency 200 MHz, is normally incident upon a sheet of silver.

i. Show that the transmitted electric field just inside the sheet is given by

$$E_t = E_i \left(\frac{2\eta_s}{\eta_s + \eta_0} \right) \, ,$$

where η_s is the impedance of the silver and η_0 is the impedance of vacuum. [40]

ii. If the silver sheet has a thickness of 1×10^{-5} m, calculate the electric field amplitude of the wave when it emerges on the other side of the sheet. (Assume $\eta_0 = 377 \, \Omega$ and $\eta_s = 5.06 \times 10^{-3}$ at 200 MHz). [20]

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Electromagnetism formulae and vector identities

Maxwell's equations are

$$\nabla \cdot \mathbf{D} = \rho \qquad\qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad\qquad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

where the symbols have their usual meanings. In LIH media $\mathbf{D} = \epsilon_r \epsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu_r \mu_0 \mathbf{H}$.

The electromagnetic potentials and the Lorenz gauge are defined by

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla V \qquad \nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$$

Useful identities (where in these examples $\mathbf{A}$ is any vector field and V any scalar field)

$$\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) \qquad \nabla \cdot (\nabla \times \mathbf{A}) = 0 \qquad \nabla \times (\nabla V) = 0$$

$$\oint \mathbf{A} \cdot d\mathbf{S} = \int (\nabla \cdot \mathbf{A})\, d^3 r \qquad\qquad \oint \mathbf{A} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

The Div and Curl operators in spherical polar coordinates (r, θ, ϕ) are given by

$$\nabla . \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

The elemental surface area and volume in spherical polar coordinates

$$dS = r^2 \sin\theta \, d\theta \, d\phi \qquad\qquad dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$$