**The Handbook of Mathematics, Physics and Astronomy Data is provided**

KEELE UNIVERSITY

EXAMINATIONS, 2012/13

Level II

Thursday 17th January 2013, 16:00 – 18:00

PHYSICS/ASTROPHYSICS

PHY-20006

QUANTUM MECHANICS

**Candidates should attempt ALL of PART A and TWO questions from PART B.**

**PART A yields 40% of the marks, PART B yields 60%.**

**A sheet of useful formulae can be found on page 9.**

**NOT TO BE REMOVED FROM THE EXAMINATION HALL**

**PART A        Answer all TEN questions**

A1 Give an expression for the probability of observing a particle in a thin shell of radius $r$ and infinitesimal width $dr$ for a particle with a spherically symmetric wave function $\Psi(r, t)$. [4]

A2 Sketch the following function and give *two* reasons why it cannot be part of a realistic wave function ($a > 0$ is a real constant).

$$f(x) = \begin{cases} e^{ax} & x < 0 \\ ax & x \geq 0 \end{cases}$$

[4]

A3 Calculate the value of $A$ for the following wave function and explain your method.

$$\Psi(x, t) = A\,\mathrm{sech}(x)\,e^{-i\omega t}$$

You may use the following integral without proof in your answer.

$$\int \mathrm{sech}^2(x)\,dx = \tanh(x) + C$$

**N.B.** $\tanh(x) \to \pm 1$ for $x \to \pm\infty$. [4]

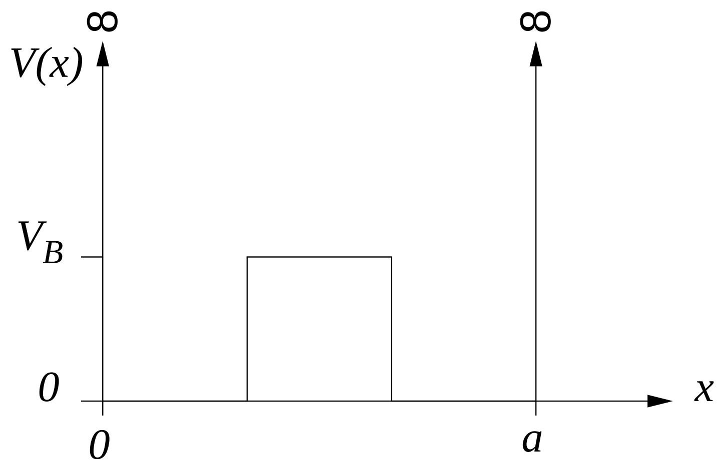
*/Cont'd*

A4 State *three* differences between the predictions of classical phys and the predictions of quantum mechanics for the properties of a particle in the semi-infinite square well potential,

$$V(x) = \begin{cases} \infty & x < 0 \\ -V_0 & 0 \leq x \leq a \\ 0 & x > a \end{cases}$$

[4]

A5 Sketch the energy eigenfunctions for the ground state and first excited state of a particle trapped in the region $0 < x < a$ by the potential shown below assuming that the energy of the particle is $E < V_B$ in both cases.




[4]

A6 Calculate the expectation value $\langle E \rangle$ for the energy of a particle with the wavefunction

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\psi_1(x)e^{-iE_1t/\hbar} - \frac{1}{\sqrt{2}}\psi_2(x)e^{-iE_2t/\hbar},$$

where $E_n = \sqrt{n}\,\text{eV}$ is the energy for the state $\psi_n$. [4]

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A7 Show that the wave function $\Psi(x,t) = e^{-i(kx+wt)}$ is an eigenfunct of the momentum operator, $\hat{p}$, and state the eigenvalue. [4]

A8 The wave function for 2 identical particles in a 1-dimensional potential is $\Psi(x_1, x_2)$. State and explain the possible values of $\Psi(x_2, x_1)$. [4]

A9 List all possible values for the magnitude of the total angular momentum, $J$, for an electron in the 2p state of the hydrogen atom. Give your answers in units of $\hbar$. [4]

A10 A particle has the wave function

$$\Psi(x,t) = \frac{a^{\frac{3}{2}}}{\sqrt{2}}\, x\, e^{-ax/2} e^{-i\omega t}, \qquad x > 0.$$

Calculate an approximate correction to the energy of the particle if the potential is perturbed by a field $V'(x) = \epsilon x$, where $\epsilon = 0.02$ eV/nm and $a = 1\,\text{nm}^{-1}$. You may use the following integral without proof in your answer.

$$\int_0^\infty r^n e^{-ar} dr = \frac{n!}{a^{n+1}}$$

[4]

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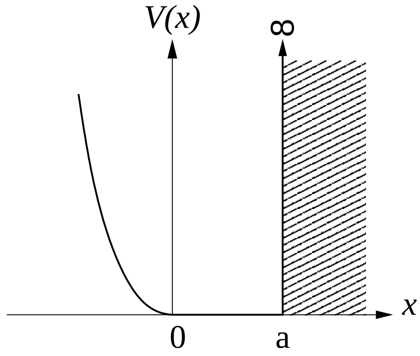
**PART B Answer TWO out of FOUR questions**

B1 A particle of mass $m$ is trapped in the following potential:

$$V(x) = \begin{cases} \frac{1}{2}kx^2 & x < 0 \\ 0 & 0 \leq x \leq a \\ \infty & x > a \end{cases}$$




For some values of $k$ it is possible to write the solution of the time independent Schrödinger equation as

$$\psi(x) = \begin{cases} \psi_A = Ae^{-\alpha x^2} & x < 0 \\ \psi_B = B\cos(\beta x) + C\sin(\beta x) & 0 \leq x \leq a \\ 0 & x > a \end{cases}$$

(a) Apply the appropriate boundary condition at $x = 0$, and hence show that $B = A$ and $C = 0$. [6]

(b) Apply the appropriate boundary condition at $x = a$ and hence derive the possible values of $\beta$. [5]

(c) Show that $\psi_B$ is a solution of the time independent Schrödinger equation, and hence derive an expression for the energy, $E$, in terms of $a$. [8]

(d) Discuss the behaviour of $\frac{\partial \psi}{\partial x}$ at $x = a$ [5]

(e) Describe and explain what the solutions of the time independent Schrödinger equation would look like in the alternative case $k \to \infty$. [6]

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B2 The energy of a particle with mass $m$ in a 2-dimensional harmo[nic] oscillator potential

$$V(x, y) = \frac{1}{2} m\omega^2 (x^2 + y^2) = \frac{1}{2} kr^2$$

is given by

$$E_{n_x, n_y} = (n_x + n_y + 1)\hbar\omega \qquad \begin{array}{l} n_x = 0, 1, 2, \ldots \\ n_y = 0, 1, 2, \ldots \end{array}$$

The angular momentum operator for a 2-dimensional system is

$$\hat{L_2} = \hat{x}\hat{p_y} - \hat{y}\hat{p_x},$$

where $\hat{p_x} = -i\hbar\frac{\partial}{\partial x}$, and similarly for $\hat{p_y}$.

(a) Write down the energy in units of $\hbar\omega$ for the first three energy levels. State the degeneracy and the values of $n_x$ and $n_y$ for each energy level. [6]

(b) Show that $Y(x, y) = y - ix$ is an eigenfunction of the operator $\hat{L_2}$. State the values of the expectation value $\langle L_2 \rangle$ and the uncertainty $\Delta L_2$ for the particle in this state. [8]

(c) Show that the commutator for $\hat{L_2}$ and $\hat{x}$ has the value $[\hat{L_2}, \hat{x}] = i\hbar y$. State briefly what your result implies for the observed values of $L_2$ and $x$. [10]

(d) Give a physical reason why eigenfunctions of the $L_2$ operator can be used to find a solution to the time independent Schrödinger equation for this potential. [6]

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B3 The energy of a particle with mass $m$ in the harmonic oscilla
potential $V(x) = \frac{1}{2}kx^2$ is given by

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \qquad n = 0, 1, 2, \ldots,$$

where $\omega = \sqrt{k/m}$. The energy eigenfunction for the first excited state is

$$\psi_1(x) = A_1 \left(\frac{x}{a}\right) e^{-x^2/2a^2},$$

where $a$ is a constant.

(a) Show that $\psi_1$ has definite parity and state its value. [4]

(b) Sketch the position probability distribution function for the state $\psi_1$. [4]

(c) The spectrum of $H^{35}Cl$ shows spectral features due to changes in vibration state spaced equally in frequency by $8.66\times10^{13}$ Hz. Calculate the bond strength, $k$, for this molecule. State any assumptions you have made in your calculation. [8]

(d) Describe how the energy eigenfunctions $\psi_j$, $j = 0, 1, 2\ldots$, for this potential can be used to represent a time-dependent wave function $\Psi(x, t)$ given an initial state $\Psi(x, 0)$. [8]

(e) Discuss briefly whether the following two statements are consistent with each other.

- The momentum, $p$, of a particle with kinetic energy $E_1$ is given by $p^2 = 2mE_1$.
- The expectation value of the momentum for the particle described by $\psi_1(x)$ is $\langle p\rangle = 0$.

[6]

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B4 The wave functions for an electron in a simple model of the hydrogen atom have the form

$$\Psi_{n,\ell,m_\ell}(r,\theta,\phi,t) = \frac{u(r)}{r} Y_{\ell,m_\ell}(\theta,\phi) e^{-iEt/\hbar}.$$

(a) State the physical quantity most closely associated with each of the quantum numbers $n$, $\ell$ and $m_\ell$, and state the possible values for each quantum number for an electron in a 3p state. [6]

(b) The function $u(r)$ is a solution of the radial Schrödinger equation for the effective potential

$$V_e(r) = \frac{\ell(\ell+1)\hbar^2}{2m_e r^2} - \frac{e^2}{4\pi\epsilon_0 r}.$$

Explain the origin of the two terms in this equation for the effective potential. [4]

(c) With the aid of a labelled diagram, describe the main features and results of the Stern-Gerlach experiment. Explain how this experiment shows that the eigenfunctions $\Psi_{n,\ell,m_\ell}$ do not give a complete description for the properties of an electron in a hydrogen atom. [12]

(d) The 3p state of hydrogen is split into two energy levels split by about 0.05 meV. Explain the origin of this *fine structure* in the energy spectrum of hydrogen. [8]

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## Quantum Mechanics formulae

## Time independent Schrödinger equation

$$\hat{H}\psi = \left[-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\right]\psi = E\psi$$