The Handbook of Mathematics, Physics and Astronomy Data is provided

KEELE UNIVERSITY

EXAMINATIONS, 2012/13

Level I

Tuesday 14th May 2013, 16.00-18.00

PHYSICS/ASTROPHYSICS

PHY-10020

OSCILLATIONS AND WAVES

Candidates should attempt ALL of PART A and ONE question from each of PARTS B and C.

PART A yields 40% of the marks, PART B yields 30%, PART C yields 30%

NOT TO BE REMOVED FROM THE EXAMINATION HALL

PART A Answer all TEN questions

A1 Write down the differential equation that defines simple harmonic oscillation, and its general solution. State two physical conditions that must be satisfied for simple harmonic motion to occur in a mechanical system. [4]

A2 An object is in simple harmonic motion with angular frequency $\omega = 6.3\ \mathrm{s}^{-1}$. At time $t = 0$, the object is $x(0) = 3.3$ cm away from its equilibrium position and has velocity $\dot{x}(0) = 36\ \mathrm{cm\ s}^{-1}$. Calculate the value of the phase constant for this motion. [4]

A3 A particle of mass 100 g executes simple harmonic motion about $x = 0$ with frequency 0.5 Hz. At a certain instant, its kinetic energy is $K = 7.5 \times 10^{-3}$ J and its potential energy is $U = 5.0 \times 10^{-3}$ J. Find the maximum speed of the particle and the amplitude of the oscillation. [4]

A4 A particular damped and driven harmonic oscillator has the equation of motion (for all quantities in SI units)

$$0.75\,\ddot{x} + 0.96\,\dot{x} + 0.48\,x = 1.44\cos(2.56\,t) .$$

Determine whether the *transient* is underdamped, critically damped or overdamped in this case. Give a representative sketch showing the main physical features of the *transient* motion. [4]

A5 A lightly damped harmonic oscillator with natural angular frequency ω_0 is driven by an external force of the form $F(t) = F_0\cos(\omega_e t)$. Write down the general form for the *steady-state* displacement $x_p(t)$ of the oscillator, and sketch the dependence of the steady-state amplitude on the driving angular frequency. [4]

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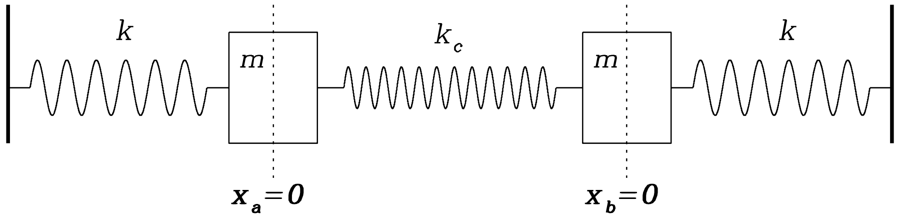
A6 Two identical blocks on identical springs, coupled by a third spr are illustrated here in equilibrium:



Give sketches indicating how the blocks move in the *normal modes* of this system. What is the defining feature of a normal mode? [4]

A7 The displacement of particles in a harmonic wave travelling along the x-axis is (for x and y in metres, and t in seconds):

$$y(x,t) \;=\; 0.005\,\sin\left(0.2\pi\,x \;+\; 40\pi\,t \;+\; \pi/12\right)\;.$$

Calculate the wavelength of this wave, and the period of simple harmonic oscillation for any individual particle in the wave. [4]

A8 Verify that the function

$$y(x,t) \;=\; 2\,\sin(2x+1)\cos(3t-4)$$

is a solution of the one-dimensional wave equation. In what direction does this wave propagate? [4]

A9 A piano wire of length $L = 0.66$ m is put under tension, with both ends fixed, such that transverse waves travel along the wire with speed $v = 330$ m s^{-1}. Calculate the wavelength λ and the frequency f of the $n = 3$ resonant *standing* wave on this wire. [4]

A10 Two identical speakers at positions $x = 0$ and $x = 20$ m face each other and emit sound waves in phase, each with equal wavelength $\lambda = 4$ m and equal volume (intensity). Find all the positions x where the total volume is maximised along a straight line between the two speakers. Justify your answer. [4]

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PART B Answer ONE out of TWO questions

B1 (a) A simple pendulum with a length of 60 cm swings with an angular amplitude of $\theta_{\text{max}} = 12°$ from the vertical. Calculate the period of this pendulum. How much time does it take to move from an angular displacement of $\theta = -6°$ directly to $\theta = +6°$? [10]

(b) A particle of mass m is in simple harmonic motion about an equilibrium position $x = 0$, with amplitude A and angular frequency ω.

i. Use the work-energy theorem to show that the potential energy of the particle (relative to the equilibrium value) is $U = \frac{1}{2}m\omega^2 x^2$. [6]

ii. Write the potential and kinetic energies of the particle as functions of *time*, and thus show that the total mechanical energy is a constant. [6]

iii. Prove that kinetic energy equals potential energy at the positions $x = \pm A/\sqrt{2}$. [4]

iv. Sketch the kinetic, potential, and total energies of the particle, as functions of x. [4]

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B2 The displacement x of a harmonic oscillator of mass m, which has a natural angular frequency ω_0 and is subjected to a damping force $F_{\text{damp}} = -b\dot{x}$, must be given by one of the following three functions of time:

$$
\begin{aligned}
&\text{(I)} && x(t) = A_0\, e^{-bt/2m} \sin(\omega t + \phi_0) \quad \textit{with} \;\; \omega \equiv \sqrt{\omega_0^2 - b^2/4m^2} \\
&\text{(II)} && x(t) = e^{-bt/2m}\,(C_1\, t + C_2) \\
&\text{(III)} && x(t) = e^{-bt/2m}\left(B_1\, e^{qt} + B_2\, e^{-qt}\right) \quad \textit{with} \;\; q \equiv \sqrt{b^2/4m^2 - \omega_0^2}
\end{aligned}
$$

(a) A block of mass $m = 2$ kg is attached to a spring with force constant $k = 8$ N m^{-1} and damping constant $b = 8$ kg s^{-1}. At time $t = 0$, the block has position $x(0) = 0.4$ m and velocity $\dot{x}(0) = 0.6$ m s^{-1}.

i. Determine whether this system is overdamped or underdamped or critically damped. Thus, which one of equations (I)–(III) above describes the motion of the block? [4]

ii. Use the initial conditions given to determine both the position and the velocity of the block at any time t. [14]

iii. Sketch $x(t)$ for this system. [4]

(b) Suppose the damping constant in the system of part (a) can be changed while all other parameters and initial conditions are kept the same. State which of equations (I)–(III) would apply to the block, and sketch and describe how its motion would change qualitatively, if the damping constant were

i. $b = 4$ kg s^{-1}. [4]

ii. $b = 16$ kg s^{-1}. [4]

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PART C Answer ONE out of TWO questions

C1 (a) A string with uniform linear mass density μ is stretched along the x-axis and kept under a constant tension F. A transverse harmonic wave travels along the string at speed $v = \sqrt{F/\mu}$, causing a displacement $y(x,t)$ at position x at time t. The kinetic energy per unit length in the string is given by

$$\frac{dK}{dx} = \frac{1}{2}\,\mu\left(\frac{\partial y}{\partial t}\right)^2$$

and the potential energy per unit length is given by

$$\frac{dU}{dx} = \frac{1}{2}\,F\left(\frac{\partial y}{\partial x}\right)^2 .$$

i. Write down a general form for the wave function $y(x,t)$ in terms of the wavenumber k and the angular frequency ω. What do the partial derivatives $\partial y/\partial t$ and $\partial y/\partial x$ represent, physically? [4]

ii. Prove that $dK/dx = dU/dx$. [8]

(b) A particular travelling wave has the function

$$y(x,t) = x\,e^{2x-4t} - 2t\,e^{2x-4t} .$$

Show that

$$\frac{\partial^2 y}{\partial t^2} + 8\,\frac{\partial y}{\partial t} + 16\,y = 0 .$$

Thus, what kind of oscillation drives this wave?

Use the one-dimensional wave equation to infer the phase speed of the wave. In what direction does the wave travel? [18]

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C2 (a) Two harmonic waves travelling along the x-axis combine to produce the standing wave,

$$y(x,t) \; = \; 2A \, \sin(kx) \cos(\omega t) \; .$$

Derive the allowed wavelengths λ_n if this is a resonant standing wave confined to $0 \leq x \leq L$, with a *node* at $x = 0$ and an *anti-node* at $x = L$. Sketch the wave function at $t = 0$ for each of the two lowest harmonics in this case. [12]

(b) Two coherent sources, S_1 and S_2, emit harmonic waves with the same amplitude A, intensity I_0, angular frequency ω, wavenumber k, and phase constant ϕ_0. These waves interfere at a point P, which is a distance x_1 from source S_1 and a distance x_2 from source S_2.

i. Derive an expression for the total wave function at P, and from this show that the total intensity at P is given by

$$I_{\text{tot}}(P) \; = \; 4 \, I_0 \cos^2 \left[\frac{k(x_1 - x_2)}{2} \right] \; . \qquad [12]$$

ii. In terms of the wavelength λ, for what values of the path difference at P is the total intensity equal to the original I_0? [6]