The Handbook of Mathematics, Physics and Astronomy Data is provided

KEELE UNIVERSITY

EXAMINATIONS, 2011/12

Level III

Tuesday 17th January 2012, 13.00-15.00

PHYSICS

PHY-30012

ELECTROMAGNETISM

Candidates should attempt to answer THREE questions.

A sheet of useful formulae and vector identities can be found on the last page.

NOT TO BE REMOVED FROM THE EXAMINATION HALL

1. An electric field in free space has the form

$$\mathbf{E} = \sin(kx - \omega t)\,\hat{\mathbf{j}} + \cos(kx - \omega t)\,\hat{\mathbf{k}} \quad \mathrm{V\,m^{-1}},$$

where the symbols have their usual meanings.

(a) Show that this field is consistent with Gauss's law in vacuum. [10]

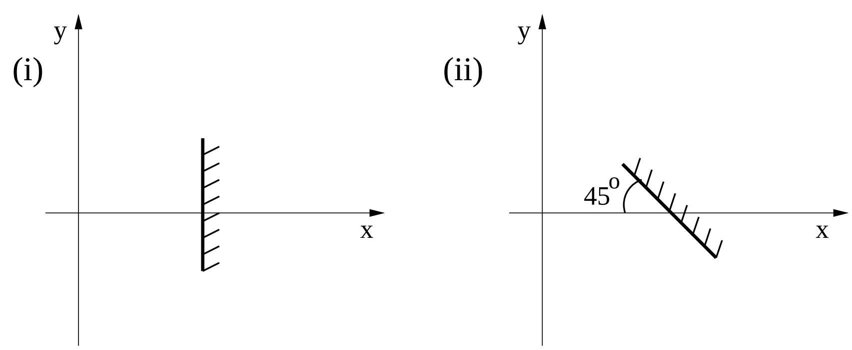
(b) Use Maxwell's equations to calculate the associated time-dependent magnetic field. [20]

(c) Show that the ratio of the E-field and B-field amplitudes is given by ω/k. [10]

(d) Show that **E** and **B** are perpendicular. [10]

(e) Give a definition of the Poynting vector in physical terms and calculate its numerical value for these electromagnetic fields. [25]

(f) Calculate the vector force exerted by these fields on a $1\,\mathrm{cm}^2$ square of perfectly reflective material for each of the two scenarios illustrated in the diagram below. [25]



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2. Using the *electric dipole approximation*, the time-depende netic field of an oscillating magnetic dipole is given by

$$\mathbf{B} = -\hat{\boldsymbol{\theta}}\, \frac{\mu_0\, \omega^2\, m_0}{4\pi\, c^2\, r} \sin\theta\, \cos[\omega(t - r/c)]\,,$$

where r is the distance from the dipole, the magnetic dipole moment is defined by $\mathbf{m} = m_0 \sin(\omega t)\, \hat{\mathbf{z}}$, and the other symbols have their usual meanings.

(a) Explain what is meant by the *electric dipole approximation.* [10]

(b) Using your knowledge of transverse electromagnetic waves, write down, with justification, an expression for the associated electric field. [20]

(c) Calculate an expression for the *time-averaged* Poynting vector. [15]

(d) Show that the total power lost by electromagnetic radiation from the system is

$$P = \frac{\mu_0\, \omega^4\, m_0^2}{12\pi\, c^3}\,.$$

[20]

(e) A highly magnetized neutron star of radius 10^4 m and mass of 3×10^{30} kg rotates with a period of 0.033 s, and has a magnetic dipole moment of magnitude 10^{28} A m^2 that is perpendicular to its rotation axis.

i. Explain how the neutron star could be treated as an *oscillating* magnetic dipole in this case and show that the *electric dipole approximation* discussed in part (a) is valid. [25]

ii. Hence, assuming that the neutron star loses rotational energy solely by the power emitted as magnetic dipole radiation, estimate an approximate spin-down timescale. [10]

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3. (a) An electromagnetic wave $\mathbf{E} = E_0 \sin(kx - \omega t)\, \hat{\mathbf{k}}$ is incid
an ionized plasma.

i. Explain how this scenario leads to Thomson scattering. [10]

ii. Show that the time-averaged square of the acceleration of an electron in the plasma is

$$< \ddot{r}^2 >= \frac{e^2\, E_0^2}{2 m_e^2} \,,$$

where e and m_e are the electron charge and mass. Justify any assumptions you make. [30]

(b) In a nuclear fusion experiment the electron density in the fusion plasma is estimated using scattered light. A laser beam with electric field $\mathbf{E} = 10^{10} \sin(10^7\, x - 3 \times 10^{15}\, t)\, \hat{\mathbf{k}}\ \ \mathrm{V\,m^{-1}}$ illuminates a plasma volume of $10^{-9}\,\mathrm{m}^3$. A camera with effective collecting area $1\,\mathrm{cm}^2$ measures the scattered light $1\,\mathrm{m}$ from the plasma, where the laser beam, plasma and camera are in the x, y plane.

i. Using your results from part (a), show that accelerated electrons in the plasma move non-relativistically. [15]

ii. If the camera collects $10^{-5}\,\mathrm{W}$ of scattered radiation, estimate the number density of electrons in the plasma. [30]

iii. The camera is rotated so that the laser beam, plasma and camera are in the x, z plane. State how the scattered light measurement changes, and explain in physical terms why. [15]

[The power scattered into unit solid angle Ω by one electron is

$$\frac{dP}{d\Omega} =< N > \frac{e^4}{16\pi^2\, \epsilon_0^2\, m_e^2\, c^4} \sin^2\theta \,,$$

where $< N >$ is the time-averaged Poynting vector magnitude of the incoming wave and θ is the angle between the electric field polarization of the incoming wave and the scattering direction.]

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4. (a) In an LIH medium, Ampère's law can be written as

$$\nabla \times \mathbf{B} = \mu_r \, \mu_0 \, \mathbf{J} + \mu_r \, \mu_0 \, \epsilon_r \, \epsilon_0 \, \frac{\partial \mathbf{E}}{\partial t} \, ,$$

where the symbols have their usual meanings.

i. Explain what physical conditions the acronym LIH refers to. [15]

ii. If the medium is also a *conductor* with conductivity σ, then show that

$$\nabla^2 \mathbf{B} = \mu_r \, \mu_0 \, \sigma \, \frac{\partial \mathbf{B}}{\partial t} + \mu_r \, \mu_0 \, \epsilon_r \, \epsilon_0 \, \frac{\partial^2 \mathbf{B}}{\partial t^2} \, .$$

[20]

(b) The solutions to the equation in a(ii) may be waves of the form $\mathbf{B} = B_0 \exp[i(kx - \omega t)] \, \hat{\mathbf{j}}$, where ω is an angular wave frequency and the magnitude of the wave-vector, k, is given by

$$k^2 = i \, \mu_r \, \mu_0 \, \sigma \, \omega + \mu_r \, \mu_0 \, \epsilon_r \, \epsilon_0 \, \omega^2 \, .$$

Describe the nature of this solution in two limiting cases: where $\sigma \ll \epsilon_r \, \epsilon_0 \, \omega$ and where $\sigma \gg \epsilon_r \, \epsilon_0 \, \omega$. Your answer should include a discussion of any dependence of the wave amplitude on x and the wave velocity and wavelength compared to their vacuum values. [45]

(c) A nuclear submarine communicates using radio waves of frequency 100 Hz. Justifying any assumptions that you make, estimate to what depth the submarine must rise in order to communicate with a surface station [For sea water you may assume that $\mu_r = 1$, $\epsilon_r = 70$ and $\sigma = 5 \, \mathrm{S \, m^{-1}}$.] [20]

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5. (a) Discuss whether Maxwell's equations and electromagne
are consistent with Special Relativity. Your discussion sh
include which quantities are invariant in different reference fram
and which are not. [30]

(b) Starting from Maxwell's equations in vacuum, show that the electric field obeys a wave equation of the form

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

and identify c with a combination of physical constants. [20]

(c) Explain how Maxwell's equations are modified to treat the propagation of electromagnetic waves in a *neutral*, *non-conducting*, LIH medium and show that the wave speed is changed. [15]

(d) Use the boundary conditions for electric fields at an interface between two media, and the example of electromagnetic waves normally incident on such an interface, to argue that the wave frequency is unchanged in a neutral, non-conducting medium. [15]

(e) A Young's double-slit interference experiment is carried out in (i) a vacuum and (ii) immersed in a non-conducting, non-magnetic, neutral fluid. For case (i), the first maximum in the interference pattern occurs at an angle of 10° to the incident light; in case (ii) the first maximum is shifted to an angle of 7°. Explain this result and calculate the dielectric susceptibility, χ_e, of the fluid. [20]

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Electromagnetism formulae and vector identities

Maxwell's equations are

$$\nabla \cdot \mathbf{D} = \rho \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

where the symbols have their usual meanings. In LIH media $\mathbf{D} = \epsilon_r \epsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu_r \mu_0 \mathbf{H}$.

The electromagnetic potentials and the Lorenz gauge are defined by

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla V \qquad \nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$$

Useful identities (where in these examples $\mathbf{A}$ is any vector field and V any scalar field)

$$\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) \qquad \nabla \cdot (\nabla \times \mathbf{A}) = 0 \qquad \nabla \times (\nabla V) = 0$$

$$\oint \mathbf{A} \cdot d\mathbf{S} = \int (\nabla \cdot \mathbf{A})\, d^3 r \qquad \oint \mathbf{A} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

The Div and Curl operators in spherical polar coordinates (r, θ, ϕ) are given by

$$\nabla . \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

The elemental surface area and volume in spherical polar coordinates

$$dS = r^2 \sin\theta \, d\theta \, d\phi \qquad dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$$