The Handbook of Mathematics, Physics and Astronomy Data is provided

KEELE UNIVERSITY

EXAMINATIONS, 2011/12

Level II

Thursday 12th January 2012, 09.30 – 11:30

PHYSICS/ASTROPHYSICS

PHY-20006

QUANTUM MECHANICS

Candidates should attempt ALL of PART A and TWO questions from PART B.

PART A yields 40% of the marks, PART B yields 60%.

A sheet of useful formulae can be found on page 8.

NOT TO BE REMOVED FROM THE EXAMINATION HALL

PART A Answer all TEN questions

A1 Give a physical interpretation of the wave function Ψ in terms of the observed position of the particle and explain how this leads to the concept of a normalised wavefunction. [4]

A2 Explain the following concepts and give an example of a physical system that demonstrates each concept.

- Wave-particle duality.
- Quantisation of energy.

[4]

A3 Calculate the expectation value for the position, $\langle x \rangle$, of a particle with the normalized wave function

$$\Psi(x,t) = \sqrt{2\pi}\, e^{-\pi x}\, e^{-i\omega t} \qquad x > 0.$$

You may use the following integral without proof in your answer.

$$\int_0^\infty r^k \exp(-\alpha r)\, dr = \frac{k!}{\alpha^{k+1}}$$

[4]

A4 State *three* differences between the predictions of classical physics and the predictions of quantum mechanics for the properties of a particle in a harmonic oscillator potential, $V(x) = \frac{1}{2}kx^2$. [4]

A5 Sketch the energy eigenfunction, ψ_1 and ψ_2, for the ground state and first excited state of the finite square-well potential

$$V(x) = \begin{cases} 0 & x < -a \\ -V_0 & -a \le x \le a \\ 0 & x > a \end{cases}$$

A6 Calculate the expectation value $\langle E \rangle$ and uncertainty ΔE for the energy of a particle with the wavefunction

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\psi_1(x)e^{-iE_1 t/\hbar} + \frac{1}{\sqrt{2}}\psi_2(x)e^{-iE_2 t/\hbar},$$

where $E_1 = 1\,\mathrm{eV}$ and $E_2 = 2\,\mathrm{eV}$. [4]

A7 Estimate the natural line width in Hz for a transition from an energy level with a lifetime $\delta t = 10^{-8}\,\mathrm{s}$. [4]

A8 Explain why identical quantum particles in the same region of space are also indistinguishable. [4]

A9 List all possible values of J, the magnitude of the total angular momentum, for an electron with orbital angular momentum quantum number $\ell = 2$. Give your answers in units of $\hbar$. [4]

A10 Explain the origin of “fine-structure” in the spectrum of the hydrogen atom. [4]

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PART B **Answer TWO out of FOUR questions**

B1 A particle of mass m is trapped in the following potential:

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 \leq x \leq a \\ \infty & x > a \end{cases}$$

(a) Show that the solutions of the time independent Schrödinger equation are

$$\psi_n(x) = \begin{cases} 0 & & x < 0 \\ \sqrt{\frac{2}{a}}\sin(n\pi x/a) & n = 1, 2, 3, \ldots, & 0 \leq x \leq a \\ 0 & & x > a \end{cases}$$

and so derive an expression for the energy of the particle in terms of a and m. [20]

(b) What is the expectation value for x in this case? Justify your answer. (Hint: No calculation required.) [3]

(c) Write down the wavefunction, $\Psi(x,t)$, for this particle in terms of a and m [3]

(d) Discuss briefly whether the following two statements are consistent with each other.

- The momentum of a particle with kinetic energy E and mass m is given by $p^2 = 2mE$.
- The expectation value of the momentum for the particle with the eigenfunction $\psi_n(x)$ is $\langle p \rangle = 0$.

[4]

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B2 Consider a particle with the wave function $\Psi(x,t) = \psi(x)e^{-i\omega t}$ wh

$$\psi(x) = \begin{cases} 0 & x \leq -1 \\ A\left[1 + \cos(\pi x)\right] & -1 < x < 1 \\ 0 & x \geq 1 \end{cases}$$

(a) Normalize this wavefunction. [8]

(b) Show that the ground state has definite parity and state its value. [4]

(c) Calculate the uncertainty in the observed position, Δx. [8]

(d) Show that ψ has the required mathematical properties for a valid wave function at the boundaries $x = \pm 1$. [5]

(e) Discuss whether $\Psi(x,t)$ can be a valid wave function for a particle in a harmonic oscillator potential if it is not one of the energy eigenfunctions, $\psi_n(x)$. [5]

You may use the following standards integrals without proof in your answers.

$$\int \left[1 + \cos(x)\right]^2\, dx = \frac{1}{4}\left[6x + 8\sin(x) + \sin(2x)\right] + C$$

$$\int_{-1}^{1} x^2[1 + \cos(\pi x)]^2\, dx = 1 - \frac{15}{2\pi^2}$$

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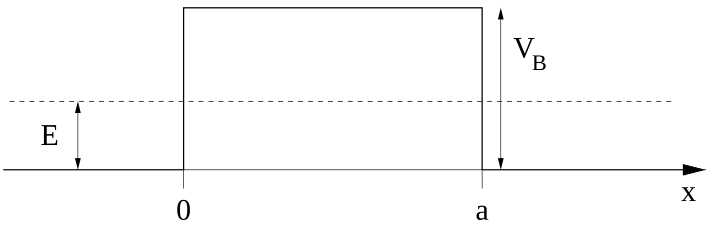
B3 Consider a particle with mass m and energy E incident on a bar with height V_B and width a, such that $E < V_B$.



(a) Show that the energy eigenfunction

$$\psi_1 = A_I e^{ikx} + A_R e^{-ikx},$$

is a solution of the time-independent Schrödinger equation for the region $x < 0$ and hence derive an expression for k in terms of E. [10]

(b) What is the physical interpretation of the quantity $R = \frac{|A_R|^2}{|A_I|^2}$? [3]

(c) State an expression for the energy eigenfunction in the region $x > a$ in terms of an amplitude A_T. Explain your answer. [6]

(d) Give one example of a physical process that demonstrates or exploits the fact that $T = \frac{|A_T|^2}{|A_I|^2} > 0$. State the physical origin and approximate value of the barrier potential, V_B, for your example. [6]

(e) Discuss what happens to the value of T in the case that the mass m is very large. [5]

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B4 The wave functions for an electron in a simple model of the hydrogen atom have the form

$$\Psi_{n,\ell,m_\ell}(r,\theta,\phi,t) = \frac{u(r)}{r} Y_{\ell,m_\ell}(\theta,\phi) e^{-iEt/\hbar}.$$

The radial eigenfunction for an electron in a hydrogen atom in the 2p state is

$$u(r) = \frac{1}{\sqrt{24a_0}} \left(\frac{r}{a_0}\right)^2 e^{-r/2a_0},$$

where a_0 is the Bohr radius.

(a) State the physical quantity most closely associated with each of the quantum numbers n, ℓ and m_ℓ, and state the possible values for each quantum number for an electron in a 2p state. [6]

(b) Calculate the expectation value, $\langle r \rangle$, for an electron in the 2p state. You may use the following standard integral without proof in your answer.

$$\int_0^\infty r^k \exp(-\alpha r)\, dr = \frac{k!}{\alpha^{k+1}}$$

[8]

(c) With the aid of a sketch, explain why $\langle r \rangle$ is different from the most probable observed value of r for the electron. [4]

(d) With the aid of a labelled diagram, describe the main features of the Stern-Gerlach experiment. Explain how this experiment shows that the eigenfunctions Ψ_{n,ℓ,m_ℓ} do not give a complete description for the properties of an electron in a hydrogen atom.

[12]

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Quantum Mechanics formulae

Time independent Schrödinger equation

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left[E - V(x)\right]\,\psi = 0$$