

**The Handbook of Mathematics, Physics and
Astronomy Data is provided**

KEELE UNIVERSITY

EXAMINATIONS, 2010/11

Level III

Wednesday 12th January 2011, 09.30-11.30

ASTROPHYSICS

PHY-30012

ELECTROMAGNETISM

Candidates should attempt to answer THREE questions.

**A sheet of useful formulae and vector identities can be found
on the last page.**

NOT TO BE REMOVED FROM THE EXAMINATION HALL

1. (a) Starting from Ampère's law in differential form, show that in the case of a vacuum, a wave equation of the form

$$\nabla^2 \mathbf{B} = \frac{1}{v^2} \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

can be obtained, and calculate the wave speed, v . [30]

- (b) Show that the magnetic field given by

$$\mathbf{B} = 10^{-5} \cos(10z - \omega t) \hat{i} \text{ T}$$

is a solution to the equation in part (a) and hence calculate the angular frequency ω . [30]

- (c) Calculate the associated time-dependent electric field by using Maxwell's equations in differential form. [20]

- (d) Calculate the time-averaged power absorbed by a perfectly black cube with sides of 2 m (aligned with the cartesian coordinate axes) that is placed into this electromagnetic field. [20]

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2. (a) The solutions to the inhomogeneous wave equation for the vector potential at position \mathbf{r} and time t , caused by current densities at position \mathbf{r}' , is given by

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' ,$$

where t' is the retarded time.

- i. Explain what is meant by *the retarded time*. [10]
 - ii. Explain what is meant by the electric dipole approximation, when dealing with time-variable current densities and, using an example, discuss how it makes $\mathbf{A}(\mathbf{r}, t)$ easier to calculate. [20]
- (b) Electromagnetic waves from an antenna of length 1 m have an electric field given by

$$\mathbf{E} = \frac{0.1}{r} \sin \theta \cos \theta \sin(3 \times 10^4 t - 10^{-4} r) \hat{\phi} \text{ V m}^{-1} ,$$

where r, θ, ϕ are the usual spherical polar coordinate system.

- i. Using your knowledge of the properties of electromagnetic waves, write down, with reasons, an expression for the magnetic field of the wave. [20]
- ii. Show that the electric dipole approximation is justified in this case. [10]
- iii. Calculate the time-averaged Poynting vector as a function of θ and hence determine how much power is emitted by the antenna in all directions. [40]

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3. An electromagnetic wave $\mathbf{E} = E_i \sin(kz - \omega t)\hat{i}$ (where k and ω have their usual meanings) is travelling in a vacuum and is incident at right angles upon an interface with a non-conducting non-magnetic medium of refractive index n .

(a) How does the speed of the wave and the ratio E/H change for a wave *in the medium*, where E and H are the magnitudes of the E-field and H-field respectively? [15]

(b) Using an appropriate form of Maxwell's equations, argue that the total E-field and total H-field of this wave must be continuous across the boundary between vacuum and medium. [35]

(c) Hence show that the reflected E-field amplitude, E_r , is related to the incident E-field amplitude, E_i , by

$$E_r = E_i \frac{(1 - n)}{(1 + n)}$$

[25]

(d) The incident wave has a wavelength of 500 nm and carries a flux of 10 W m^{-2} . If the reflected flux is 0.4 W m^{-2} , calculate the refractive index of the medium and the wavelength of the *transmitted wave*. [25]

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4. (a) State two differences and two similarities between *Mie scattering* and *Rayleigh scattering* of electromagnetic radiation. [20]

(b) Larmor's formula for the power radiated by an accelerating charge is given by

$$P = \frac{q^2 \langle a^2 \rangle}{6\pi\epsilon_0 c^3},$$

where q is the charge and $\langle a^2 \rangle$ is the time average of the square of its acceleration.

i. Write down an expression for the acceleration of a charged particle by the fields of an incident plane-polarised electromagnetic wave, stating and justifying any assumptions that you make. [15]

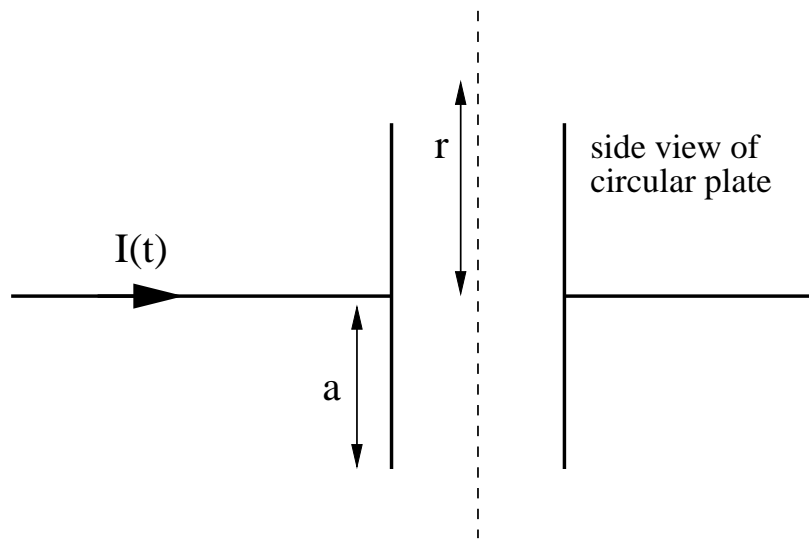
ii. Hence explain why, in any ionised gas, it is the free electrons that are the major sources of scattered light. [15]

iii. Calculate a numerical value for the scattering cross section of an electron and explain over what range of electromagnetic wave frequencies it will be valid. [25]

(c) A narrow laser beam of flux 1 W m^{-2} is fired at a small, closed cell containing completely ionised sodium gas. A radiation detector of collecting area 1 cm^2 situated 1 m away from the cell, and at right angles to the line of the incident laser beam, records a scattered light power of 10^{-9} W . Calculate the mass of the illuminated sodium gas. [25]

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5. (a) i. Starting from Ampère's law in differential form, $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$, derive the integral form of the equation using Stokes's theorem. Identify what is meant by the *displacement current*. [10]
- ii. Briefly explain why the displacement current term in Ampère's law is necessary for the existence of electromagnetic waves. [20]
- (b) A current $I(t)$ flows into a plane, parallel capacitor with circular plates of radius a (see diagram).
- i. Calculate expressions for the displacement current between the capacitor plates and for the magnetic field as a function of radial distance, r , from the axis of the capacitor along the dashed line shown in the diagram. Your answer should include points with $r < a$ and $r > a$. [40]
- ii. Explain how and why your answers would change if the current was the same, but the space between the capacitor plates were filled with a non-magnetic insulator with a dielectric susceptibility of 1.0. [15]
- iii. How would your answers change if the space were filled with a magnetic material with magnetic susceptibility of 1000 and dielectric susceptibility of zero? [15]



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Electromagnetism formulae and vector identities

Maxwell's equations are

$$\begin{aligned}\nabla \cdot \mathbf{D} &= \rho & \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0 & \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\end{aligned}$$

where the symbols have their usual meanings. In LIH media $\mathbf{D} = \epsilon_r \epsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu_r \mu_0 \mathbf{H}$.

The electromagnetic potentials and the Lorenz gauge are defined by

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla V \quad \nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} = 0$$

Useful identities (where in these examples \mathbf{A} is any vector field and V any scalar field)

$$\begin{aligned}\nabla \times (\nabla \times \mathbf{A}) &= -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) & \nabla \cdot (\nabla \times \mathbf{A}) &= 0 & \nabla \times (\nabla V) &= 0 \\ \oint \mathbf{A} \cdot d\mathbf{S} &= \int (\nabla \cdot \mathbf{A}) d^3r & \oint \mathbf{A} \cdot d\mathbf{l} &= \int (\nabla \times \mathbf{A}) \cdot d\mathbf{S}\end{aligned}$$

The Div and Curl operators in spherical polar coordinates (r, θ, ϕ) are given by

$$\begin{aligned}\nabla \cdot \mathbf{A} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} \\ \nabla \times \mathbf{A} &= \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r \sin \theta \hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & r A_\theta & r \sin \theta A_\phi \end{vmatrix}\end{aligned}$$

The elemental surface area and volume in spherical polar coordinates

$$dS = r^2 \sin \theta d\theta d\phi \quad dV = r^2 \sin \theta dr d\theta d\phi$$