The Handbook of Mathematics, Physics and Astronomy Data is provided

KEELE UNIVERSITY

EXAMINATIONS, 2010/11

Level III

Wednesday 12th January 2011, 09.30-11.30

PHYSICS

PHY-30004

ELECTROMAGNETISM

Candidates should attempt to answer THREE questions.

A sheet of useful formulae and vector identities can be found on the last page.

NOT TO BE REMOVED FROM THE EXAMINATION HALL

1. (a) Starting from Ampère's law in differential form, show
the case of a vacuum, a wave equation of the form

$$\nabla^2 \mathbf{B} = \frac{1}{v^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}$$

can be obtained, and calculate the wave speed, v. [30]

(b) Show that the magnetic field given by

$$\mathbf{B} = 10^{-5}\cos(10z - \omega t)\hat{\imath}\ \ \mathrm{T}$$

is a solution to the equation in part (a) and hence calculate the angular frequency ω. [30]

(c) Calculate the associated time-dependent electric field by using Maxwell's equations in differential form. [20]

(d) Calculate the time-averaged power absorbed by a perfectly black cube with sides of 2 m (aligned with the cartesian coordinate axes) that is placed into this electromagnetic field. [20]

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2. (a) The solutions to the inhomogeneous wave equation for the vector potential at position **r** and time t, caused by current densities at position $\mathbf{r}'$, is given by

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\, d^3\mathbf{r}' \,,$$

where t' is the retarded time.

i. Explain what is meant by *the retarded time.* [10]

ii. Explain what is meant by the electric dipole approximation, when dealing with time-variable current densities and, using an example, discuss how it makes $\mathbf{A}(\mathbf{r},t)$ easier to calculate. [20]

(b) Electromagnetic waves from an antenna of length 1 m have an electric field given by

$$\mathbf{E} = \frac{0.1}{r}\sin\theta\cos\theta\,\sin(3\times10^4\,t - 10^{-4}\,r)\,\hat{\phi}\ \ \mathrm{V\,m^{-1}}\,,$$

where r, θ, ϕ are the usual spherical polar coordinate system.

i. Using your knowledge of the properties of electromagnetic waves, write down, with reasons, an expression for the magnetic field of the wave. [20]

ii. Show that the electric dipole approximation is justified in this case. [10]

iii. Calculate the time-averaged Poynting vector as a function of θ and hence determine how much power is emitted by the antenna in all directions. [40]

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3. An electromagnetic wave $\mathbf{E} = E_i \sin(kz - \omega t)\hat{\imath}$ (where t bols have their usual meanings) is travelling in a vacuum a incident at right angles upon an interface with a non-conductin non-magnetic medium of refractive index n.

(a) How does the speed of the wave and the ratio E/H change for a wave *in the medium*, where E and H are the magnitudes of the E-field and H-field respectively? [15]

(b) Using an appropriate form of Maxwell's equations, argue that the total E-field and total H-field of this wave must be continuous across the boundary between vacuum and medium. [35]

(c) Hence show that the reflected E-field amplitude, E_r, is related to the incident E-field amplitude, E_i, by

$$E_r = E_i \frac{(1-n)}{(1+n)}$$

[25]

(d) The incident wave has a wavelength of 500 nm and carries a flux of 10 $\mathrm{W\,m^{-2}}$. If the reflected flux is 0.4 $\mathrm{W\,m^{-2}}$, calculate the refractive index of the medium and the wavelength of the *transmitted wave*. [25]

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4. (a) State two differences and two similarities between *scattering* and *Rayleigh scattering* of electromagnetic radia [2

(b) Larmor's formula for the power radiated by an accelerating charge is given by

$$P = \frac{q^2 \left\langle a^2 \right\rangle}{6\pi\epsilon_0 c^3} ,$$

where q is the charge and $\left\langle a^2 \right\rangle$ is the time average of the square of its acceleration.

i. Write down an expression for the acceleration of a charged particle by the fields of an incident plane-polarised electromagnetic wave, stating and justifying any assumptions that you make. [15]

ii. Hence explain why, in any ionised gas, it is the free electrons that are the major sources of scattered light. [15]

iii. Calculate a numerical value for the scattering cross section of an electron and explain over what range of electromagnetic wave frequencies it will be valid. [25]

(c) A narrow laser beam of flux $1\,\mathrm{W\,m^{-2}}$ is fired at a small, closed cell containing completely ionised sodium gas. A radiation detector of collecting area $1\,\mathrm{cm^2}$ situated $1\,\mathrm{m}$ away from the cell, and at right angles to the line of the incident laser beam, records a scattered light power of $10^{-9}\,\mathrm{W}$. Calculate the mass of the illuminated sodium gas. [25]

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5. (a) i. Starting from Ampère's law in differential form, d… integral form of the equation using Stokes's theorem… identify what is meant by the *displacement current*. [1…

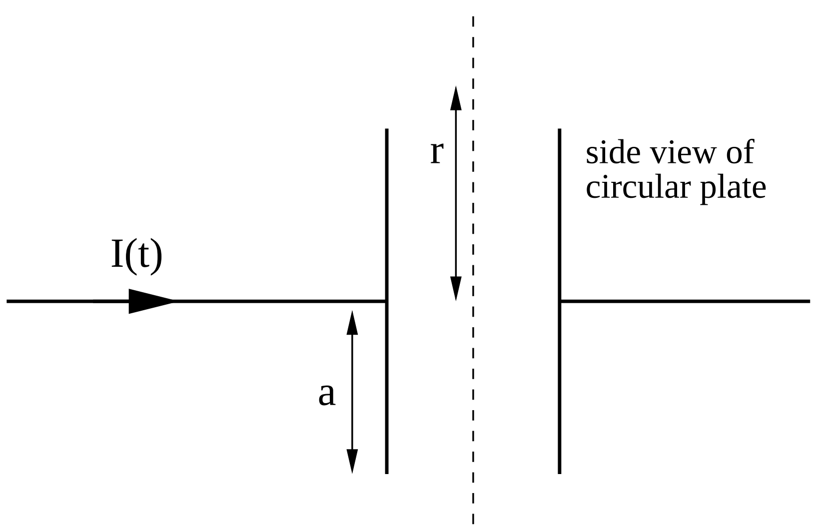
ii. Briefly explain why the displacement current term in Ampère's law is necessary for the existence of electromagnetic waves. [20]

(b) A current $I(t)$ flows into a plane, parallel capacitor with circular plates of radius a (see diagram).

i. Calculate expressions for the displacement current between the capacitor plates and for the magnetic field as a function of radial distance, r, from the axis of the capacitor along the dashed line shown in the diagram. Your answer should include points with $r < a$ and $r > a$. [40]

ii. Explain how and why your answers would change if the current was the same, but the space between the capacitor plates were filled with a non-magnetic insulator with a dielectric susceptibility of 1.0. [15]

iii. How would your answers change if the space were filled with a magnetic material with magnetic susceptibility of 1000 and dielectric susceptibility of zero? [15]



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Electromagnetism formulae and vector identities

Maxwell's equations are

$$\nabla \cdot \mathbf{D} = \rho \qquad\qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad\qquad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

where the symbols have their usual meanings. In LIH media $\mathbf{D} = \epsilon_r \epsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu_r \mu_0 \mathbf{H}$.

The electromagnetic potentials and the Lorenz gauge are defined by

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla V \qquad \nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$$

Useful identities (where in these examples $\mathbf{A}$ is any vector field and V any scalar field)

$$\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) \qquad \nabla \cdot (\nabla \times \mathbf{A}) = 0 \qquad \nabla \times (\nabla V) = 0$$

$$\oint \mathbf{A} \cdot d\mathbf{S} = \int (\nabla \cdot \mathbf{A})\, d^3 r \qquad\qquad \oint \mathbf{A} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

The Div and Curl operators in spherical polar coordinates (r, θ, ϕ) are given by

$$\nabla . \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

The elemental surface area and volume in spherical polar coordinates

$$dS = r^2 \sin\theta\, d\theta\, d\phi \qquad\qquad dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$$