The Handbook of Mathematics, Physics and Astronomy Data is provided

KEELE UNIVERSITY

EXAMINATIONS, 2010/11

Level II

Friday 21st January 2011, 09.30-11.30

PHYSICS/ASTROPHYSICS

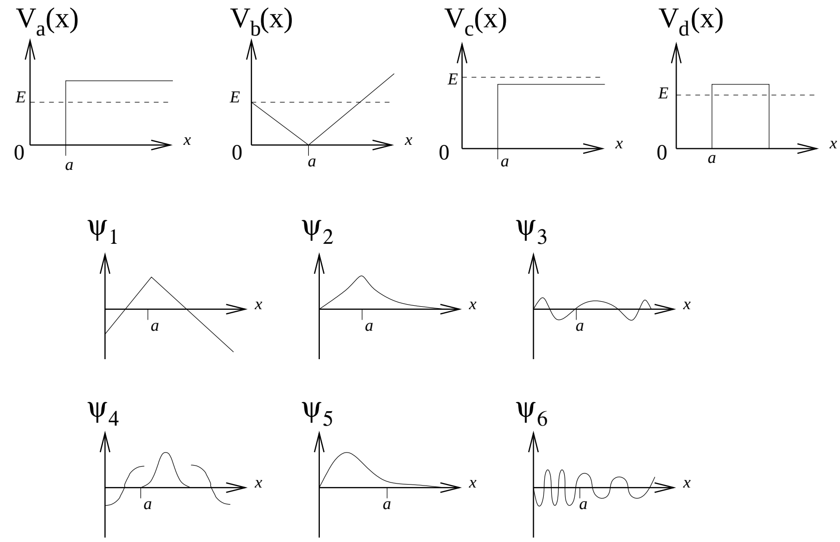
PHY-20006

QUANTUM MECHANICS

Candidates should attempt to answer FOUR questions.

NOT TO BE REMOVED FROM THE EXAMINATION HALL

1. (a) For each of the potentials $\mathrm{V_a}, \ldots, \mathrm{V_d}$, state which of the fu tions $\psi_1, \psi_2, \ldots$, is the most likely solution of the time-independe Schrödinger equation for the particle with energy E.



(A larger version of this figure appears on the last page of this exam paper.) [4 × 10]

(b) The wave function for a particle restricted to the region $x > 0$ is

$$\psi(x) = \begin{cases} \psi_1 = A(1 - \cos(x)) & 0 \leq x \leq 3\pi/2 \\ \psi_2 = Be^{-x} & x > 3\pi/2 \end{cases}$$

i. State two boundary conditions that the wave function must satisfy at $x = 3\pi/2$. [10]

ii. Show that this is a valid wave function if $A = Be^{-3\pi/2}$. [30]

(c) Show that the following wave function has definite parity and state its value.

$$\psi(x) = A\sin(x)/x$$ [20]

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2. (a) State and explain the relation between the observed position of a particle, x, and its wave function, $\Psi(x,t)$, i.e., the Born interpretation of the wave function. [20]

(b) A particle is restricted to the region $x > 0$ in a 1-dimensional potential for which the wave function in the ground state is

$$\Psi(x,t) = Ax^{\frac{1}{4}}e^{-ax}e^{-iEt/\hbar}.$$

i. Explain what is meant by the *ground state* and its relation to the concept of *zero-point energy.* [10]

ii. State and explain how the value of the normalization constant, A, is calculated. [10]

iii. Show that the value of A in this case is $\left(32a^3/\pi\right)^{\frac{1}{4}}$. [30]

You may use the following integral without proof in your answers.

$$\int_0^\infty x^{\frac{1}{2}}e^{-x}dx = \frac{\sqrt{\pi}}{2}$$

(c) An experiment is devised to study a large number of non-interacting particles all in the same quantum state, $\psi(x)$. State how the values of the following observables can be predicted from $\psi(x)$. (You are **not** required to calculate the values).

i. The average position of the particles. [10]

ii. The most likely value of the position. [10]

iii. The average momentum of the particles. [10]

/Cont'd

3. The 1-dimensional time-independent Schrödinger equation for a particle of mass m in a potential $V(x)$ is

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left[E - V(x)\right]\,\psi = 0$$

A particle is restricted to the region $0 < x < a$ by the potential

$$V(x) = \begin{cases} \infty & x \leq 0 \\ 0 & 0 < x < a \\ \infty & x \geq a \end{cases}$$

(a) State and explain the value of the solution ψ in the regions $x \leq 0$ and $x \geq a$ in this case. [20]

(b) Show that the particle has energy states

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

where $n = 1, 2, 3, \ldots$. [60]

(c) State the main differences between the predictions of classical physics and quantum mechanics for the behaviour of the particle in this potential. [20]

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4. A quantum particle in the harmonic oscillator potential $V(x) = \frac{1}{2}$
has energy levels

$$E_n = \left(n + \frac{1}{2}\right) \hbar\omega, \qquad n = 0, 1, 2, \ldots,$$

where $\omega = \sqrt{k/m}$ and m is the mass of the particle.

(a) Sketch the probability distribution for the observed position of the particle in the state $\psi_1(x) = A_1 \left(\frac{x}{a}\right) e^{-x^2/2a^2}$ where a is a constant. [20]

(b) The bond strength in a $^{12}C^{18}O$ molecule is $k = 1908\,\mathrm{Nm}^{-1}$. Calculate the wavelength of the photon emitted due to the transition between the vibrational states $n = 3 \rightarrow 1$. [35]

(c) Describe briefly how the energy eigenfunctions for this potential ψ_i, $i = 0, 1, 2 \ldots$, can be used to represent a time-dependent wave function $\Psi(x, t)$. [25]

(d) Discuss briefly whether the following two statements are consistent with each other.

- The momentum, p, of a particle with kinetic energy E_1 is given by $p^2 = 2mE_1$.
- The expectation value of the momentum for the particle described by $\psi_1(x)$ is $\langle p \rangle = 0$.

[20]

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5. The wave functions for an electron in a simple model of the hydrogen atom have the form

$$\Psi_{n,\ell,m_\ell}(r,\theta,\phi,t) = \frac{u(r)}{r} Y_{\ell,m_\ell}(\theta,\phi) e^{-iEt/\hbar}.$$

The radial eigenfunction for an electron in a hydrogen atom in the 1s state is

$$u(r) = \frac{2}{\sqrt{a_0}} \left(\frac{r}{a_0}\right) e^{-r/a_0},$$

where a_0 is the Bohr radius.

(a) For each of the following quantum numbers, give the physical quantity most closely associated with this quantum number and state the possible values for this quantum number for an electron in a 1s state.

i. n

ii. ℓ

iii. m_ℓ

[20]

(b) Show that the expectation value, $\langle r \rangle$, for an electron in the 1s state is $\frac{3}{2}a_0$. [20]

You may use the following integral without proof in your answer.

$$\int_0^\infty r^k \exp(-\alpha r)\, dr = \frac{k!}{\alpha^{k+1}}$$

(c) Outline the main features of the Stern-Gerlach experiment. Explain how this experiment shows that the eigenfunctions ψ_{n,ℓ,m_ℓ} do not give a complete description for the properties of an electron in a hydrogen atom. [40]

(d) State and explain the boundary conditions that are imposed on the function $u(r)$. [20]

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6. Consider two identical particles, p and q, which can be observed the same region with positions x_p and x_q, respectively. The particles can be observed in one of two states, ψ_1 and ψ_2.

   (a) Give an expression for the two-particle state $\psi(x_1, x_2)$ in terms of $\psi_1(x_p)$, $\psi_2(x_p)$, $\psi_1(x_q)$and $\psi_2(x_q)$, assuming that the particles are

      i. fermions,

      ii. bosons.

   [20]

   (b) What is the interpretation of the state $\psi(x_1, x_2)$ in terms of the observed positions of the particles, x_p and x_q? [20]

   (c) What is the value of $\psi(x_1, x_2)$ for a pair of fermions in the case $\psi_1 = \psi_2$? Explain briefly why this leads to an understanding of the chemical properties of atoms. [40]

   (d) In classical physics, identical particles can be distinguished given their initial position and momentum. Explain why this is not the case for identical quantum particles, i.e, why quantum particles are indistinguishable. [20]

Figure for question 1(a).