KEELE UNIVERSITY

DEGREE EXAMINATIONS 2008

Level 3 (PRINCIPAL COURSE)

January 2009, 9:30 - 11:30

Astrophysics

PHY-30004

ELECTROMAGNETISM AND RADIATION

Candidates should attempt to answer THREE questions.

A sheet of useful formulae for this examination is included on the last page. Tables of general physical and mathematical data may be obtained from the invigilator.

/Cont'd

1. An electric field in a vacuum is given by the expression

$$\mathbf{E} = \sin(ky)\cos(\omega t)\hat{k}\,,$$

where k is the magnitude of the wave vector and ω is an angular frequency. Assume that the electric potential is zero and that no time-independent fields are present.

   (a) Explain how this electric field satisfies Gauss's law. [10]

   (b) Use Maxwell's equations to calculate the associated magnetic field. [30]

   (c) Calculate the magnetic vector potential and show that $\mathbf{B} = \nabla \times \mathbf{A}$ [30]

   (d) Calculate the Poynting vector and the time averaged power per unit area carried by these fields. Explain your result in physical terms. [30]

2. (a) Using Ampére's law in differential form, explain why a *scalar* magnetic potential (analagous to the electric potential) would only work for situations with time-independent electric fields and no currents. [20]

   (b) A long solenoid of radius a, with N turns per metre and carrying a constant current I is aligned along the z-axis. The magnetic vector potential is given by

$$\mathbf{A} = \frac{\mu_0 N I}{2}(-y\hat{i} + x\hat{j}) \qquad \text{or} \qquad \mathbf{A} = \frac{\mu_0 N I a^2}{2(x^2+y^2)}(-y\hat{i} + x\hat{j})\,,$$

   for points inside or outside the solenoid respectively.

   (i) Calculate the B-field inside the solenoid and show that the B-field outside the solenoid is zero. [40]

   (ii) Explain why these B-fields are consistent with Maxwell's equations in differential form. [20]

   (c) The current through the solenoid is decreased with time such that

$$I = I_0 \exp(-\lambda t)\,.$$

   Calculate expressions for the time-dependent electric field both inside and outside the solenoid. [20]

/Cont'd

3. (a) A block of dielectric material, which can be considered linear, isotropic and homogeneous, is held between the plates of a capacitor which is charged until there is a total conduction charge of $\pm Q_C$ on the plates.

(i) Define what is meant by the polarization field, **P**, inside the dielectric. [10]

(ii) Explain what are meant by the terms linear, isotropic and homogeneous in this context. [15]

(iii) Draw a diagram of the capacitor; mark on **E**, **P** and representative charges. Explain why the electric field between the capacitor plates is smaller than it would be for a capacitor with similarly charged plates but with the dielectric replaced by a vacuum. [25]

(b) In a medium, the first of Maxwell's equations becomes

$$\nabla \cdot \mathbf{D} = \rho_C \, ,$$

where **D** is the displacement field and ρ_C is conduction charge density. The capacitor in part (a) has square plates of side 10^{-2} m, that hold a charge of 10^{-9} C and contains a nylon dielectric block with a dielectric susceptibility of 2.7.

(i) Use Gauss's divergence theorem to calculate the magnitude of the displacement field between the capacitor plates. [20]

(ii) Calculate the strengths of the electric field and polarization field in the nylon dielectric. [20]

(iii) Calculate the total charge on one surface of the dielectric block. [10]

/Cont'd

4. (a) Explain what is meant by the electric dipole approximation. [10]

(b) The magnetic field due to an electric dipole oscillating along the z-axis is given by

$$\mathbf{B} = -\hat{\phi}\,\frac{\mu_0 l I_0 \omega \sin\theta}{4\pi r c}\,\sin[\omega(t - r/c)]\,,$$

where l is the length of the dipole, I_0 is the current amplitude, ω is the angular frequency of oscillation, r is the radial distance from the dipole, θ is the polar angle from the z-axis and $\hat{\phi}$ is the azimuthal unit vector in spherical polar coordinates.

(i) For two points that are equidistant from the oscillator but with $\theta = \pi/4$ and $\theta = 3\pi/4$, make a sketch in the $x - z$ plane showing the *directions* of the magnetic field, the electric field and the Poynting vector at those points, choosing an instant of time when $\mathbf{B}$ is not zero. [20]

(ii) Sketch two graphs, showing how the B-field amplitude and the time-averaged Poynting vector vary with θ, for $0 < \theta < \pi$. Carefully mark the values of the minima and maxima of each graph. [30]

(c) The oscillator is covered by a totally reflective cap formed from a section of a spherical shell with radius R, centred on the oscillator. The z-axis passes through the centre of the cap and its rim is defined by the equations $r = R$, $\theta = \pi/6$ radians.

Calculate an expression for the time-averaged total upward force (along the z-axis) exerted on the cap due to radiation from the oscillator. [40]

/Cont'd

5. (a) Explain what is meant by Thomson scattering. [10]

(b) Write down an equation of motion for a free electron interacting with a plane-polarized electromagnetic wave. State why the influence of the magnetic field can be neglected for non-relativistic situations. [20]

(c) Larmor's formula for the power radiated by an accelerated electron can be written

$$P = \frac{e^2 \langle \ddot{\mathbf{r}}^2 \rangle}{6\pi\epsilon_0 c^3}$$

where $\langle \ddot{\mathbf{r}}^2 \rangle$ is the time-average of the square of the acceleration and e is the electron charge.

Using Larmor's formula and your result from part (b), show that the Thomson scattering cross-section is given by

$$\chi_{\mathrm{T}} = \frac{e^4}{6\pi m_e^2 \epsilon_0^2 c^4} ,$$

where m_e is the electron mass. [30]

(d) Certain types of active galactic nuclei (AGN) are deduced to consist of a very bright, central illuminating source that is obscured by an edge-on ring of dark material. Light from the hidden nucleus can still be observed if it is scattered into the line of sight from clouds of completely ionized material above and below the ring.

Using your knowledge of the properties of Thomson scattered light and an appropriate diagram, explain why and under what circumstances the scattered light from an obscured AGN may be linearly polarized. [40]

/Cont'd

/Cont'd

Electromagnetism formulae and vector identities

Maxwell's equations are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

where the symbols have their usual meanings.

The electromagnetic potentials and the Lorenz gauge are defined by

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla V \qquad \nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$$

The general solutions to the inhomogeneous wave equations are given by (where $\mathbf{A}$ and V are the magnetic vector potential and electric scalar potential respectively).

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|} \, d^3 r' \qquad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|} \, d^3 r'$$

Useful identities (where in these examples $\mathbf{A}$ is any vector field and V any scalar field)

$$\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) \qquad \nabla \cdot (\nabla \times \mathbf{A}) = 0 \qquad \nabla \times (\nabla V) = 0$$

$$\oint \mathbf{A} \cdot d\mathbf{S} = \int (\nabla \cdot \mathbf{A}) \, d^3 r \qquad \oint \mathbf{A} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

The Del, Div and Curl operators in spherical polar coordinates (r, θ, ϕ) are given by

$$\nabla V = \hat{r} \frac{\partial V}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial V}{\partial \theta} + \hat{\phi} \frac{1}{r \sin\theta} \frac{\partial V}{\partial \phi}$$

$$\nabla . \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

The elemental surface area and volume in spherical polar coordinates are

$$dS = r^2 \sin\theta \, d\theta \, d\phi \qquad dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$$