KEELE UNIVERSITY

DEGREE EXAMINATIONS, 2009

Level 2 (PRINCIPAL COURSE)

DAYDAY DATEDATE, 14:30 - 16:30

PHYSICS/ASTROPHYSICS

PHY-20006

QUANTUM MECHANICS

Candidates should attempt to answer FOUR questions.

Tables of physical and mathematical data may be obtained from the invigilator.

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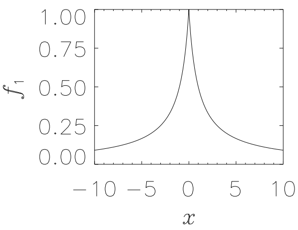
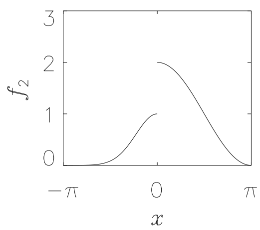
1. (a) What is a wavefunction? [10]

(b) For each of the three functions given below, state why the function is not allowable as part of a wavefunction and state all values of x at which the function is not allowable.

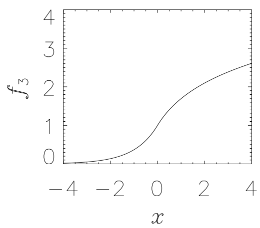
i. $$f_1(x) = \begin{cases} 1/(1-x) & (x < 0) \\ 1/(1+x) & (x \geq 0) \end{cases}$$



ii. $$f_2(x) = \begin{cases} \exp(-x^2) & (x \leq 0) \\ 1+\cos(x) & (0 < x < \pi) \\ 0 & (x \geq \pi) \end{cases}$$



iii. $$f_3(x) = \begin{cases} e^x & (x < 0) \\ 1+\log_e(1+x) & (x \geq 0) \end{cases}$$



[3 × 10]

(c) The wavefunction for a particle has the following dependance on position, x.

$$\psi(x) = \begin{cases} \psi_A = A(1+\sin x) & -\pi/2 < x < \pi \\ \psi_B = Bx^{\beta} & x \geq \pi \end{cases}$$

Show that the mathematical properties for an allowable wavefunction at $x = \pi$ require that $\beta = -\pi$. [40]

(d) Explain why a wavefunction $\Psi(x,t)$ must also be a continuous function of time, t. [20]

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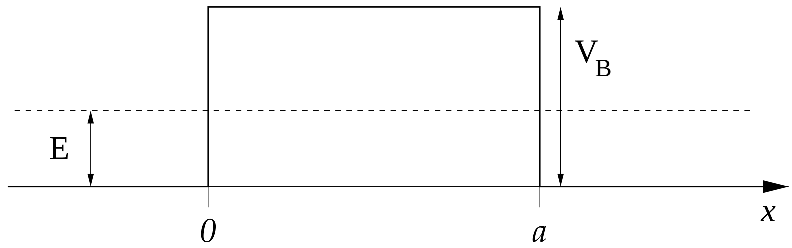
2. Consider a particle with mass m and energy E incident on a barrier with height V_B and width a such that $E < V_B$.



(a) Show that the energy eigenfunction

$$\psi_1 = A_I e^{ikx} + A_R e^{-ikx},$$

is a solution of the time independent Schrödinger equation for the region $x < 0$ if $k^2 = 2mE/\hbar^2$. [30]

(b) What is the physical interpretation of the quantity $R = \frac{|A_R|^2}{|A_I|^2}$? [10]

(c) Give an expression for the energy eigenfunction in the region $x > a$ in terms of the amplitude A_T. Explain your answer. [15]

(d) Give one example of a physical process that demonstrates or exploits the fact that $T = \frac{|A_T|^2}{|A_I|^2} > 0$. State the physical origin and approximate height of the barrier, V_B, for your example. [25]

(e) What would be the energy eigenfunction in the region $0 < x < a$ in the case of a more energetic particle with $E > V_B$? Explain your answer. [20]

The time-independent Schrödinger equation in 1-dimension is:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\,\psi = E\psi.$$

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3. A particle of mass m is located in a 'square well' of width a, where the potential V inside the well is zero and is infinite otherwise.

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 \leq x \leq a \\ \infty & x > a \end{cases}$$

The energy eigenfunctions for the particle in the region $0 \leq x \leq a$ is

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin(n\pi x/a),$$

where $n = 1, 2, 3, \ldots$.

The energy of the particle in the state ψ_n is

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}.$$

(a) What are the main differences between the predictions of classic physics and the predictions of quantum mechanics regarding the energy of the particle? [10]

(b) Show that the probability of observing the particle in the region $x < a/3$ approaches the value predicted by classical physics if n is large. [40]

(c) Give an expression for the energy eigenfunctions of a particle trapped in a cube of side a together with an expression for the energy of these states. [25]

(d) What is the largest number of electrons that can be contained in a cube of side a with energy less than or equal to $3\pi^2\hbar^2/(m_e a^2)$? Explain your answer. (Ignore any interactions between the electrons for the purposes of answering this question). [25]

You may find the following standard integral to be useful.

$$\int \sin^2 x \, dx = \frac{1}{2}(x - \sin x \cos x) + C$$

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4. The harmonic oscillator potential for a particle of mass m is

$$V(x) = \frac{1}{2}kx^2.$$

The solutions of the time-independent Schrödinger equation for this potential have energies given by the expression

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \qquad n = 0, 1, 2, \ldots,$$

where $\omega = \sqrt{k/m}$.

Consider a particle in the state described by the eigenfunction

$$\psi_1(x) = A_1 \left(\frac{x}{a}\right) e^{-x^2/2a^2},$$

where $a = \sqrt{\hbar/m\omega}$.

(a) Explain how the value of the constant A_1 is calculated. (You do not need to calculate A_1, but you should show the first step in the calculation.) [10]

(b) Show that the most probable value for the observed position of the particle with the eigenfunction $\psi_1(x)$ is $x = \pm a$. [20]

(c) Sketch the probability distribution for the observed position of the particle with the eigenfunction $\psi_1(x)$. Include an indication of the scale on the x-axis of your sketch. [10]

(d) Show that the eigenfunction ψ_1 has well defined parity and state the value of the parity. [10]

(e) The bond strength in a $^{13}C^{16}O$ molecule is k =1908 Nm^{-1}. Calculate the wavelength of the photon emitted due to the transition between the vibrational states $n = 3 \rightarrow 1$. [30]

(f) Discuss briefly whether the following two statements are consistent with each other.

- The momentum of a particle with kinetic energy E_1 is given by $p = \sqrt{2mE_1}$.
- The expectation value of the momentum for the particle described by $\psi_1(x)$ is $\langle p \rangle = 0$.

[20]

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5. The eigenfunctions for a particle in a central potential have the form

$$\psi_{n,\ell,m_\ell}(r,\theta,\phi) = R_{n,\ell}(r)Y_{\ell,m_\ell}(\theta,\phi).$$

(a) Explain the significance of the commutator relation $[\hat{L}^2, \hat{L}_z] = 0$. [10]

(b) What is the relation between the function $Y_{\ell,m_\ell}(\theta,\phi)$, the operators $\hat{L}^2$ and $\hat{L}_z$ and the quantum numbers ℓ and m_ℓ. [20]

(c) The dependence of the eigenfunctions on the azimuthal coordinate ϕ is given by

$$\psi_{n,l,m_l} \propto e^{im_l\phi}.$$

By considering the values of ψ_{n,l,m_l} at angles ϕ and $\phi + 2\pi$, show that m_l must be an integer. [15]

(d) Explain why the polar diagram for states with $\ell = 0$ is a circle. [15]

(e) Outline the main features of the Stern-Gerlach experiment. Explain how this experiment shows that the eigenfunctions ψ_{n,ℓ,m_ℓ} do not give a complete description for the properties of an electron in a hydrogen atom. [40]

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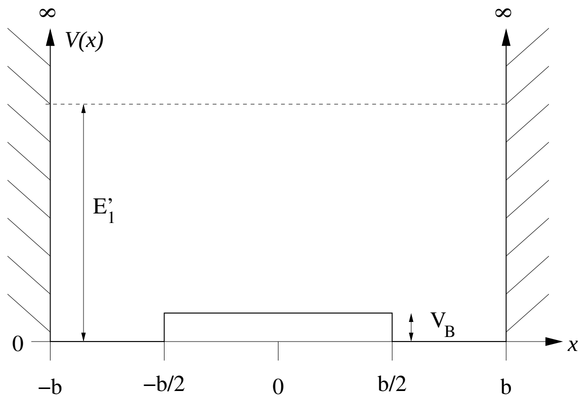
6. Consider a particle in the ground state with energy E_1' for the potential shown below. The value of the potential for $x < -b$ and $x > b$ is $V(x) = \infty$.



The step of height V_B can be considered as a perturbation to the infinite square well potential. The energy eigenfunction of the ground state for the unperturbed potential ($V_B = 0$) is

$$\psi_1 = \sqrt{\frac{1}{b}} \cos\left(\frac{\pi x}{2b}\right).$$

(a) What is the value of the wavefunction for the particle for $x > b$? Explain your answer. [10]

(b) With the aid of a sketch, describe the main differences between the unperturbed energy eigenfunctions of the ground state, ψ_1, and the perturbed energy eigenfunctions , ψ_1' for the perturbed potential ($V_B > 0$). [30]

(c) Estimate the ground state energy, E_1', for the perturbed potential in terms of E_1 and V_B. [40]

(d) Discuss whether your expression for E_1' in part (c) can be applied to the case $V_B < 0$. [20]

You may find the following standard integral to be useful.

$$\int \cos^2 x \, dx = \frac{1}{2}(x + \sin x \cos x) + C$$

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