Imperial College London

BSc/MSci EXAMINATION June 2013

This paper is also taken for the relevant Examination for the Associateship

THERMODYNAMICS AND STATISTICAL PHYSICS

For 2nd-Year Physics Students

Monday, 10th June 2013: 10:00 to 12:00

Answer all questions
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the FOUR answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in FOUR answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. (i) Show that in a reversible, constant pressure process in an ideal gas $\Delta V/V_0 = \Delta T/T_0$, where V_0 and T_0 are the initial volume and temperature.

10 kg of argon, initially at a temperature of 273 K and a pressure of 10^5 Pa, is heated so that its temperature rises to 373 K. During this process the pressure remains fixed. Assuming that argon can be treated as an ideal gas in which the molecules have a mass 6.63×10^{-26} kg, calculate $\Delta V/V_0$ and the work done by the gas during this process. [5 marks]

(ii) Given that the internal energy of an ideal gas is $U = \frac{n_d}{2} N k_B T$, where n_d is the number of degrees of freedom (you may assume this expression without proof), obtain expressions for C_V and C_P (the constant volume and constant pressure heat capacities) of an ideal gas, and show that they differ by Nk_B. Briefly explain why C_P is larger than C_V.

Argon is a monatomic gas. Calculate the heat supplied to raise the temperature of the argon in the constant pressure process discussed in part (i). [7 marks]

(iii) The volume thermal expansivity, β, and the isothermal compressibility, κ_T, are defined as:

$$\beta = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P \text{ , and } \kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$$

Show that $dV = V(\beta dT - \kappa_T dP)$.

Calculate the values of β and κ_T for an ideal gas at a temperature of 273 K and a pressure of 10^5 Pa. [5 marks]

(iv) In many situations the volume thermal expansivity of a solid can be assumed to be constant. Making this assumption show that in a constant pressure process in a solid $\Delta V/V_0 = e^{\beta \Delta T} - 1$. [5 marks]

(v) At a temperature of 273 K and a pressure of 10^5 Pa iron has the following properties:

density: $\rho = 7.87 \times 10^3$ kg m^{-3},

volume thermal expansivity: $\beta = 3.54 \times 10^{-5}$ K^{-1},

constant pressure specific heat: $c_P = 449$ J K^{-1} kg^{-1}.

A 10 kg block of iron is initially at a temperature of 273 K and a pressure of 10^5 Pa. Heat is supplied to the block as a result of which its temperature rises to 373 K. During this process the pressure remains constant. Assuming that β and c_P remain constant during this process, calculate: (a) the heat supplied, (b) $\Delta V/V_0$, and (c) the work done by the iron block.

Briefly explain why C_V and C_P differ very little for iron. [8 marks]

[Total 30 marks]

2. (i) Write down the fundamental equation of thermodynamics. Identify all the state variables in the equation, and indicate which are extensive which are intensive. [6 marks]

(ii) The Helmholtz function is defined as $F = U - TS$. Show that the work done on a system undergoing a reversible isothermal process is equal to the change in its Helmholtz function.

By writing F as a function of the appropriate pair of state variables derive equations for P and S in terms of derivatives of F, and, hence, obtain the following Maxwell relation:

$$\left(\frac{\partial P}{\partial T}\right)_V = \left(\frac{\partial S}{\partial V}\right)_T .$$

[6 marks]

(iii) Using the equations obtained in Part (ii), or otherwise, derive the following equation (the energy equation):

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P .$$

Use this equation to show that the internal energy of an ideal gas depends only on temperature. [6 marks]

(iv) A photon gas inside and in equilibrium with a closed cavity at temperature T has the following properties (which you may assume without proof): (1) the energy density, $\hat{u}$, is a function only of temperature, and, (2) $P = \hat{u}/3$.

Use the energy equation to show that $\hat{u} = \Lambda T^4$, where Λ is a constant.

The volume of the cavity is changed reversibly, while its temperature is held fixed. Show that the change in the Helmholtz function of the photon gas is

$$\Delta F = -\frac{\Lambda}{3} T^4 \Delta V .$$

[8 marks]

(v) A paper is submitted to a scientific journal reporting measurements of the macroscopic properties of a certain liquid. The author concludes that the volume and internal energy of the liquid satisfy the following two equations:

$$V = V_0\{1 + \beta_0(T - T_0) - \kappa_0(P - P_0)\} , \quad U = CT + \alpha V$$

where V_0, T_0, P_0, β_0, κ_0, C and α are all constants. The paper is rejected, the referee stating that the results violate basic thermodynamics. Is the referee right? Justify your answer. [4 marks]

[Total 30 marks]

3. Consider an elastic band, hanging with a weight m attached to the bottom. We can consider the band to be a thermodynamic system in thermal equilibrium with its environment at a temperature T. The band consists of N distinguishable massless segments, each of length a. Each segment can be aligned up or down, so the maximum length of the band is aN. The weight of the mass tends to pull the segments downwards and lengthen the band, but thermal energy can make segments change alignment so the weight can rise or fall.

(i) What kind of ensemble does this scenario correspond to? Write down the general expression for Z_1, the partition function for a single segment, and the probability p_j of it being in state j with energy ϵ_j. [5 marks]

(ii) All the segments are the same length a, so that changing the alignment of a single segment changes the overall length by $2a$. What is the difference in potential energy of the mass when an individual segment is aligned upwards or downwards? Hence write down the full expression for Z_1 in this situation . [5 marks]

(iii) Determine the probability of a segment pointing downwards or upwards and hence show that the length of the elastic band is given by $L = Na \tanh\left(\frac{amg}{k_B T}\right)$. [5 marks]

(iv) We now gently heat the elastic band. Based on your results, what do you expect to happen, and why? [5 marks]

[Total 20 marks]

4. This question concerns a Bose-Einstein gas, where you may assume that the lowest energy state, $\epsilon_0 = 0$ and you may ignore degeneracy.

(i) Write down the Bose-Einstein distribution function, $f_{BE}(E)$, defining the terms that you use. [5 marks]

(ii) The density of states of a gas is given by

$$g(\epsilon)d\epsilon = \frac{2\pi V}{h^3}(2m)^{3/2}\epsilon^{1/2}d\epsilon.$$

At very low temperatures, a significant number of particles enter the ground state so the density of states approximation does not account for them. Explain, briefly and qualitatively, why this is the case.

At these low temperatures, the chemical potential $\mu \approx 0$. Using this fact, show that the number of particles in excited states is given by

$$n' = 4.62 \times \pi V \left(\frac{2mk_B T}{h^2}\right)^{3/2}.$$

[5 marks]

(iii) Now calculate the internal energy of the cold boson gas and hence show that the constant volume heat capacity is given by

$$C_V = AT^{3/2}$$

and find an expression for A. This demonstrates that in contrast to the classical result, the heat capacity tends to zero at absolute zero for a boson gas. [10 marks]

You may use without proof the following mathematical results:

$$\int_0^\infty \frac{x^{1/2}dx}{e^x - 1} \approx 2.31$$

$$\int_0^\infty \frac{x^{3/2}dx}{e^x - 1} \approx 1.78$$

[Total 20 marks]