Vibrations & Waves Homework Sheet 2
ANSWERS
(QUESTIONS given out: Monday 24 January 2005)

1)

$$x(t) = (A + Bt)\exp(-\frac{r}{2m}t)$$

Critically damped: $\frac{r}{2m} = \sqrt{\frac{s}{m}} = \omega_0$

Therefore can write: $x(t) = (A + Bt)\exp(-\omega_0 t)$

Also:

$$m\frac{d^2x(t)}{dt^2} + r\frac{dx(t)}{dt} + sx(t) = 0$$

$$=> \frac{d^2x(t)}{dt^2} + \frac{r}{m}\frac{dx(t)}{dt} + \frac{s}{m}x(t) = 0$$

$$=> \frac{d^2x(t)}{dt^2} + 2\omega_0\frac{dx(t)}{dt} + \omega_0^2 x(t) = 0$$

Take final x(t) part of above equation:

$$\omega_0^2[x(t)] = \omega_0^2\left[(A + Bt)\exp(-\omega_0 t)\right]$$

$$= (\omega_0^2 A + \omega_0^2 Bt)\exp(-\omega_0 t)$$

Take middle first derivative part of above equation:

$$2\omega_0\left[\frac{dx(t)}{dt}\right] = 2\omega_0\left[(-\omega_0 A - \omega_0 Bt + B)\exp(-\omega_0 t)\right]$$

$$= (-2\omega_0^2 A - 2\omega_0^2 Bt + 2\omega_0 B)\exp(-\omega_0 t)$$

Take initial second derivative part of above equation:

$$\left[\frac{d^2x(t)}{dt^2}\right] = \left[(\omega_0^2 A + \omega_0^2 Bt - \omega_0 B - \omega_0 B)\exp(-\omega_0 t)\right]$$

1) continued)

$$\left[\frac{d^2x(t)}{dt^2}\right] + 2\omega_0\left[\frac{dx(t)}{dt}\right] + \omega_0^2[x(t)]$$
$$= (\omega_0^2 A + \omega_0^2 Bt - \omega_0 B - \omega_0 B)\exp(-\omega_0 t)$$
$$+(-2\omega_0^2 A - 2\omega_0^2 Bt + 2\omega_0 B)\exp(-\omega_0 t)$$
$$+(\omega_0^2 A + \omega_0^2 Bt)\exp(-\omega_0 t)$$
$$= (\omega_0^2 A - 2\omega_0^2 A + \omega_0^2 A)\exp(-\omega_0 t)$$
$$+(-\omega_0 B - \omega_0 B + 2\omega_0 B)\exp(-\omega_0 t)$$
$$+(\omega_0^2 Bt - 2\omega_0^2 Bt + \omega_0^2 Bt)\exp(-\omega_0 t)$$
$$= 0$$

All the terms with *A*, *B* and *Bt* cancel
=> solution is valid

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2)

a)i)
$$x(t) = A\exp(-\frac{r}{2m}t)\cos(\omega' t + \phi)$$
$$v(t) = A\left[-\frac{r}{2m}\exp(-\frac{r}{2m}t)\cos(\omega' t + \phi) - \omega'\exp(-\frac{r}{2m}t)\sin(\omega' t + \phi)\right]$$

ii) Evaluate at t=0 :
$$x(t=0) = A\cos(\phi)$$
$$v(t=0) = A\left[-\frac{r}{2m}\cos(\phi) - \omega'\sin(\phi)\right]$$

iii) At t= 0, x = 0 and v = v_0
$$x(t=0) = A\cos(\phi) = 0$$
$$v(t=0) = A[-\omega'\sin(\pi/2)] = A[-\omega'] = v_0$$

Therefore: $\phi = \pi/2$ and $A = -v_0/\omega'$

=> $x(t) = \frac{-v_0}{\omega'}\exp(-\frac{r}{2m}t)\sin(\omega' t)$

Note: you could solve this like part (c) below for heavily damped using the initial exponential complex form for lightly damped & taking the real part at the end

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b)i) $x(t) = (A + Bt)\exp(-\frac{r}{2m}t)$

$$v(t) = -\frac{r}{2m}A\exp(-\frac{r}{2m}t) + B\exp(-\frac{r}{2m}t) - \frac{r}{2m}Bt\exp(-\frac{r}{2m}t)$$

=> $v(t) = \left\lfloor B - \frac{r}{2m}A - \frac{r}{2m}Bt \right\rfloor \exp(-\frac{r}{2m}t)$

ii) Evaluate at t=0 :

$x(t=0) = A$

$v(t=0) = \left\lfloor B - \frac{r}{2m}A \right\rfloor$

iii) At t= 0, x = 0 and v = v_0

$x(t=0) = A = 0$

$v(t=0) = B = v_0$

Therefore:

$x(t) = v_0 t\exp(-\frac{r}{2m}t)$

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c)i)

$$x(t) = \exp(-pt)[C_1\exp(qt) + C_2\exp(-qt)]$$
$$= [C_1\exp((-p+q)t) + C_2\exp((-p-q)t)]$$

$$v(t) = [C_1(-p+q)\exp((-p+q)t) + C_2(-p-q)\exp((-p-q)t)]$$
$$= \exp(-pt)[C_1(-p+q)\exp(qt) + C_2(-p-q)\exp(-qt)]$$

ii) Evaluate at t=0 :

$x(t=0) = C_1 + C_2$

$v(t=0) = C_1(-p+q) + C_2(-p-q)$

iii) At t= 0, x = 0 and v = v_0

$x(t=0) = C_1 + C_2 = 0$ => $C_2 = -C_1$

$v(t=0) = C_1(-p+q) - C_1(-p-q) = 2qC_1 = v_0$

=> $C_1 = v_0 / 2q$

=> $x(t) = \frac{v_0}{2q}\exp(-pt)[\exp(qt) - \exp(-qt)]$

3)

i) $$LA\rho\frac{d^2x(t)}{dt^2} = -8\pi\eta L\frac{dx(t)}{dt} - 2A\rho g x(t)$$

ii) **Rearrange:** $$LA\rho\frac{d^2x}{dt^2} + 8\pi\eta L\frac{dx}{dt} + 2A\rho g x = 0$$

Substitute three variables:

$$m = LA\rho$$
$$r = 8\pi\eta L$$
$$s = 2A\rho g$$

=> $$m\frac{d^2x(t)}{dt^2} + r\frac{dx(t)}{dt} + sx(t) = 0$$

Trial solution: $x(t) = \underline{C}\exp(\underline{\alpha}t)$

Put into above Eqtn: $\underline{C}\exp(\underline{\alpha}t)\left[m\underline{\alpha}^2 + r\underline{\alpha} + s\right] = 0$

=> $m\underline{\alpha}^2 + r\underline{\alpha} + s = 0$

Two roots:

$$\underline{\alpha} = -\frac{r}{2m} \pm \sqrt{\left(\frac{r^2}{4m^2} - \frac{s}{m}\right)}$$

Let: $\underline{\alpha} = -p \pm q$

$$p = \frac{r}{2m} \qquad q = \sqrt{\left(\frac{r^2}{4m^2} - \frac{s}{m}\right)}$$

General solution

$$x(t) = \underline{C}_1\exp((-p+q)t) + \underline{C}_2\exp((-p-q)t)$$

iii)a) Lightly damped:

$$q = jq' = j\sqrt{\left(\frac{s}{m} - \frac{r^2}{4m^2}\right)}$$

General solution:

$$x(t) = \underline{C}_1\exp((-p+jq')t) + \underline{C}_2\exp((-p-jq')t)$$

Let:

$$\underline{C}_1 = \frac{B}{2}\exp(j\phi) \quad \underline{C}_2 = \frac{B}{2}\exp(-j\phi)$$

$$x(t) = \frac{B}{2}\exp(-pt)\left[\exp(j(q't+\phi)) + \exp(-j(q't+\phi))\right]$$

Real part: $x(t) = B\exp(-pt)\cos(q't+\phi)$

For U tube oscillator:

$$x(t) = B\exp\left(-\frac{4\pi\eta}{A\rho}t\right)\cos(\omega't+\phi)$$

where:

$$\omega' = \sqrt{\left(\frac{2g}{L} - \left(\frac{4\pi\eta}{A\rho}\right)^2\right)} = \sqrt{\left(\omega_0^2 - \left(\frac{4\pi\eta}{A\rho}\right)^2\right)}$$

b) Critically damped:

$$\frac{r^2}{4m^2} = \frac{s}{m}$$

Term in bracket in q zero:

=> $\underline{\alpha} = -p \pm q = -p$

=> $x(t) = (\underline{C}_1 + \underline{C}_2)\exp(-pt)$

But must have two roots =>

$$x(t) = (C + Bt)\exp(-pt)$$

For U tube oscillator:

$$x(t) = (C + Bt)\exp\left(-\frac{4\pi\eta}{A\rho}t\right)$$

c) Heavily damped:

$$x(t) = \exp(-pt)\left[C_1\exp(qt) + C_2\exp(-qt)\right]$$

For U tube oscillator:

$$x(t) = \exp\left(-\frac{4\pi\eta}{A\rho}t\right)\left[C_1\exp(qt) + C_2\exp(-qt)\right]$$

where:

$$q = \sqrt{\left(\left(\frac{4\pi\eta}{A\rho}\right)^2 - \frac{2g}{L}\right)} = \sqrt{\left(\left(\frac{4\pi\eta}{A\rho}\right)^2 - \omega_0^2\right)}$$

iv) $TE = A\rho g h^2$

For starting conditions x(t=0) = h, v(t=0) = 0

Lightly damped U tube oscillator

=> $x(t) = h\exp(-\frac{4\pi\eta}{A\rho}t)\cos(\omega' t)$

Amplitude of lightly damped SHM decays as:

$$h(t) = h(t=0)\exp(-\frac{4\pi\eta}{A\rho}t)$$

Therefore TE decays as:

$$TE(t) = A\rho g h^2 \exp(-\frac{2\pi\eta}{A\rho}t)$$

Rate of change of energy with time:

$$\frac{dTE(t)}{dt} = -\frac{2\pi\eta}{A\rho}A\rho g h^2 \exp(-\frac{2\pi\eta}{A\rho}t) = -\frac{2\pi\eta}{A\rho}TE(t)$$

Quality factor:

$$Q = 2\pi\frac{TE(t)}{-\left(\frac{dTE(t)}{dt}\frac{2\pi}{\omega'}\right)} = \frac{A\rho}{2\pi\eta}\omega'$$

v)

Light: $\left(\frac{4\pi\eta}{A\rho}\right)^2 < \frac{2g}{L}$

Critical: $\left(\frac{4\pi\eta}{A\rho}\right)^2 = \frac{2g}{L}$

Heavy: $\left(\frac{4\pi\eta}{A\rho}\right)^2 > \frac{2g}{L}$

For both water and oil: $\frac{2g}{L} = \frac{2\times 10}{1} = 20$

For water: $\left(\frac{4\pi\eta}{A\rho}\right)^2 = \left(\frac{4\pi 10^{-3}}{10^{-4}\times 10^{3}}\right)^2 = 0.0158$

For oil: $\left(\frac{4\pi\eta}{A\rho}\right)^2 = \left(\frac{4\pi\times 0.2}{10^{-4}\times 0.9\times 10^3}\right)^2 = 780$

Water lightly damped, oil heavily damped

For water: $\omega' = \sqrt{\left(\frac{2g}{L} - \left(\frac{4\pi\eta}{A\rho}\right)^2\right)} = \sqrt{(20 - 0.0158)} = 4.47 rad/s$

$$Q = \frac{A\rho}{2\pi\eta}\omega' = \frac{10^{-4}\times 10^3}{2\pi\times 10^{-3}}4.47 = 71$$

vi)

$$x(t) = \frac{P_0 A}{\omega Z_m}\sin(\omega t - \phi)$$

$$Z_m = \sqrt{\left[(8\pi\eta L)^2 + (\omega L A\rho - 2A\rho g/\omega)^2\right]}$$

(vii)

$$h = \frac{P_0 A}{\omega Z_m} = \frac{P_0 A}{\omega\sqrt{\left[(8\pi\eta L)^2 + (\omega L A\rho - 2A\rho g/\omega)^2\right]}}$$

(viii)

$$\omega_r = \sqrt{\left(\frac{2g}{L} - 2\left(\frac{4\pi\eta}{A\rho}\right)^2\right)}$$

For water: $\omega_r = \sqrt{(20 - 2(0.0158))} = 4.47 rad/s$

(ix) $Q = \frac{A\rho}{2\pi\eta}\omega' \approx \frac{A\rho}{2\pi\eta}\omega_0 = \frac{\omega_0}{\omega_2 - \omega_1}$

Bandwidth:

$$\omega_2 - \omega_1 = 2\pi\eta / A\rho$$

Range of frequencies:

$$\omega_2 = \omega_0 + \pi\eta / A\rho = \sqrt{20} + \pi 10^{-3}/0.1 = 4.504$$

$$\omega_1 = \omega_0 - \pi\eta / A\rho = \sqrt{20} - \pi 10^{-3}/0.1 = 4.441$$

SoM, VW & QP 2003 Exam Question (6)

(i) $$\mathbf{m\frac{d^2\underline{x}}{dt^2} + r\frac{d\underline{x}}{dt} + s\underline{x} = F_0 \exp(j\omega t)}$$

(ii) $$\underline{x} = \underline{A}\exp(j\omega t)$$

$$\mathbf{\frac{d\underline{x}}{dt} = j\omega \underline{A}\exp(j\omega t) = j\omega \underline{x}}$$

$$\mathbf{\frac{d^2\underline{x}}{dt^2} = -\omega^2 \underline{A}\exp(j\omega t) = -\omega^2 \underline{x}}$$

Therefore: $\mathbf{\left(-\omega^2 m + rj\omega + s\right)\underline{A}\exp(j\omega t) = F_0 \exp(j\omega t)}$

$$=> \mathbf{\underline{A} = \frac{F_0}{\left(-\omega^2 m + rj\omega + s\right)}}$$

$$\mathbf{\underline{A} = \frac{F_0}{\omega\left((s/\omega - \omega m) + jr\right)} = \frac{-jF_0}{\omega\left(r + j(\omega m - s/m)\right)}}$$

$$\mathbf{\underline{Z}_m = Z_m \exp(j\phi) = \sqrt{\left[r^2 + (\omega m - s/m)^2\right]}\exp(j\phi)}$$

$$=> \mathbf{\underline{A} = \frac{-jF_0}{\omega Z_m \exp(j\phi)} = \frac{-jF_0 \exp(-j\phi)}{\omega Z_m}}$$

where: $\mathbf{Z_m = \sqrt{\left[r^2 + (\omega m - s/\omega)^2\right]}}$

(iii) $$\mathbf{\underline{x} = \frac{-jF_0 \exp(j(\omega t - \phi))}{\omega Z_m}}$$

$$\mathbf{\underline{x} = \frac{-jF_0}{\omega Z_m}\left(\cos((\omega t - \phi)) + j\sin((\omega t - \phi))\right)}$$

$$\mathbf{\underline{x} = \frac{F_0}{\omega Z_m}\left(-j\cos((\omega t - \phi)) + \sin((\omega t - \phi))\right)}$$

Real part: $\mathbf{x = \frac{F_0}{\omega Z_m}\sin(\omega t - \phi)}$

(iv) Maximum displacement

=> when $\dfrac{F_0}{\omega Z_m}$ maximum

=> when ωZ_m a minimum

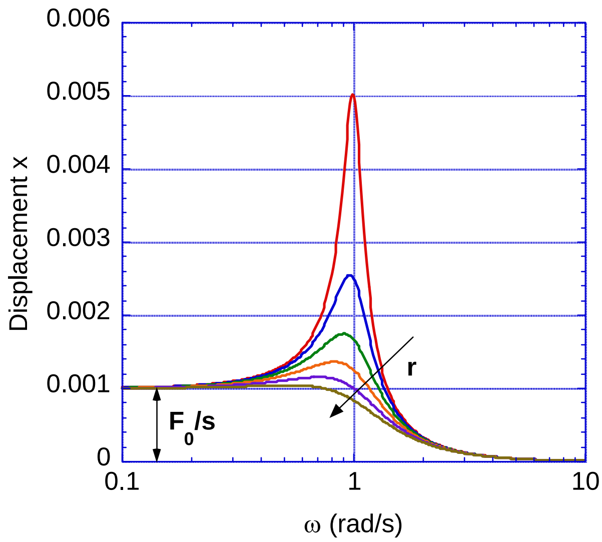
$$\omega\sqrt{\left[\mathbf{r}^2+(\omega\mathbf{m}-\mathbf{s}/\omega)^2\right]}$$
$$=\sqrt{\omega^2\left[\mathbf{r}^2+\omega^2\mathbf{m}^2+\mathbf{s}^2/\omega^2-2\mathbf{sm}\right]}$$
$$=\sqrt{\left[\omega^4\mathbf{m}^2+\omega^2\mathbf{r}^2-2\mathbf{sm}\omega^2+\mathbf{s}^2\right]}$$

Find when term in braket minimum

$$\Rightarrow \frac{\mathbf{d}}{\mathbf{d}\omega}\left[\omega^4\mathbf{m}^2+\omega^2\mathbf{r}^2-2\mathbf{sm}\omega^2+\mathbf{s}^2\right]$$
$$=\left[4\omega^3\mathbf{m}^2+2\omega\mathbf{r}^2-4\mathbf{sm}\omega\right]$$
$$=0$$

$$\left[4\omega^3\mathbf{m}^2+2\omega\mathbf{r}^2-4\mathbf{sm}\omega\right]=0$$
$$4\omega^2\mathbf{m}^2=4\mathbf{sm}-2\mathbf{r}^2$$
$$\omega^2=\frac{\mathbf{s}}{\mathbf{m}}-\frac{\mathbf{r}^2}{2\mathbf{m}}$$
$$\omega_{\mathbf{r}}=\sqrt{\omega_0^2-\frac{\mathbf{r}^2}{2\mathbf{m}}}$$

(v)



Want large **r** / small **Q** => smooth response over broad range of angular frequencies. Also no sharp resonant peak => big waves can't damage machine.

Want $\omega_0 \approx$ average angular frequency of waves => maximum power absorption.

(vi) Displacement below resonance: $\mathbf{F_0/s} = 0.5$m => $\mathbf{s} = 2000$ N/m
Resonant frequency wanted = 1 Hz.= 1 rad/s = $\boldsymbol{\omega_r}$.
Mass = 1000 kg.

$$\omega_r = \sqrt{\omega_0^2 - \frac{\mathbf{r}^2}{2\mathbf{m}^2}}$$

$$\omega_0^2 - \omega_r^2 = \frac{\mathbf{r}^2}{2\mathbf{m}^2} \quad \Rightarrow \quad \mathbf{r} = \sqrt{2\mathbf{m}^2((\frac{\mathbf{s}}{\mathbf{m}})^2 - \omega_r^2)}$$

=> $\sqrt{2(1000)^2((\frac{2000}{1000})^2 - 1^2)}$

= 2449 Ns/m

Maximum displacement: $\mathbf{x}_{\mathbf{max}} = \dfrac{\mathbf{F}_0}{\omega \mathbf{Z}_\mathbf{m}} = \dfrac{\mathbf{F}_0}{\omega\sqrt{\left[\mathbf{r}^2 + (\omega\mathbf{m} - \mathbf{s}/\omega)^2\right]}}$

$$x_{\max}(1rad/s) = \frac{1000}{1\sqrt{\left[(2449)^2 + (1000 - 2000)^2\right]}}$$

= 0.378 m