Vibrations & Waves Homework Sheet 2
QUESTIONS
(given out: Monday 24 February 2005)

Covers material in V & W Lectures 3, 4, 5 & 6

1) Using the fact that for critically damped SHM:

$$\omega_0 = \sqrt{\frac{s}{m}} = \frac{r}{2m}$$

show that:

$$x(t) = (A + Bt)\exp(-\frac{r}{2m}t)$$

is a solution of:

$$m\frac{d^2x(t)}{dt^2} + r\frac{dx(t)}{dt} + sx(t) = 0$$

2) For each of the standard solutions for the displacement x(t) of a (a) lightly damped, (b) critically damped and (c) heavily damped horizontal mass-on-a-spring, derive (i) the velocity v(t), (ii) the general values for x(t) and v(t) at t = 0, and (iii) the specific solutions for the starting conditions x(t=0) = 0 and v(t=0) = v_0.

3) Consider the U shaped tube from the last Homework Problem Sheet 1 Question 5. A liquid moving in such a tube should experience a resistive force:

$$F_{res} = -8\pi\eta L v$$

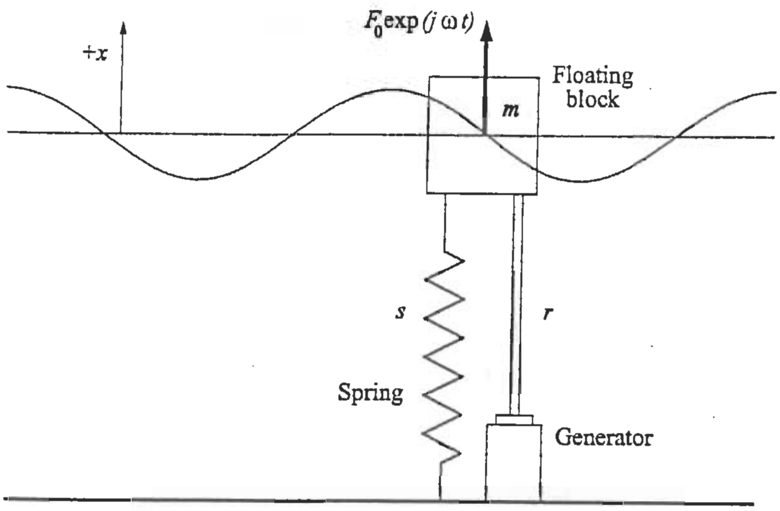
where η is the viscosity and v is the velocity of the liquid. As before the tube is tilted so that the liquid is displaced by +h on one side and –h on the other side. It is then returned instantaneously to the vertical.
(i) Write down the Equation of Motion containing the resistive force.
(ii) By substituting in the variables m, r and s to simplify the equation, derive the general solution for the vertical displacement x(t) of the liquid with time t.
(iii) By a similar substitution, derive the variation of x(t) in the (a) lightly damped, (b) critically damped and (c) heavily damped regimes.
(iv) In the last Homework Problem Sheet the variation of the potential energy PE of the oscillating liquid with x was found to be:

$$TE = A\rho g h^2$$

Using this relationship, derive the quality factor Q for the liquid.
(v) Let A = 10^{-4} m^2 and L = 1 m. For water at room temperature ρ = 10^3 kgm^{-3} and η = 10^{-3} Nsm^{-2}. For a typical engine oil at 30°C ρ = 0.9×10^3 kgm^{-3} and η = 200×10^{-3} Nsm^{-2}. For both liquids, would the oscillations be lightly, critically or heavily damped? If lightly damped, what is ω' and Q?
An air pump is attached to one end of the tube. It provides a pressure $P_0 = F_0/A$ at a driving angular frequency ω.
(vi) What is the variation with time of the vertical displacement x(t).
(vii) What is the variation of the amplitude h with driving angular frequency ω.
(viii) What is the resonant angular frequency ω_r of water in the tube.
(ix) Over what range of angular frequencies is the maximum power transferred from the pump to the water in the tube?

Exam Question, 2003 Paper

6. A machine is designed to extract energy from ocean waves arriving at the coast. A floating block of total mass m is tethered to the seabed by a spring with spring constant s. A pole extends from the block to the seabed where it drives an electrical generator which damps the block's motion with an effective mechanical resistance r. Surface water waves exert a vertical force on the block of $F_0 \exp(j\omega t)$.



(i) Write down the equation of motion for the vertical displacement $\underline{x}$ of the floating block.

[3 marks]

(ii) Show that the equation of motion has a steady-state solution of the form

$$\underline{x} = \underline{A} \exp(j\omega t) \, ,$$

where

$$\underline{A} = -\frac{jF_0 \exp(-j\phi)}{\omega Z_m}$$

and

$$Z_m = \sqrt{\left[r^2 + (\omega m - s/\omega)^2\right]} \, .$$

[5 marks]

(iii) Find the real part of the solution.

[2 marks]

(iv) Using the answer to (iii), show that the resonant frequency ω_r where maximum displacement occurs is given by

$$\omega_r = \sqrt{\omega_0^2 - \frac{r^2}{2m^2}} \, .$$

What is ω_0?

[2 marks]

(v) Sketch the variation of the maximum displacement with ω in the region surrounding ω_r. Show how this varies with r. An engineer wants an approximately constant response to any forces exerted by waves on the machine over a range of frequencies. What advice would you give?

[3 marks]

(vi) The maximum force ever exerted by the waves on the block is 1 kN. The engineer wants the block to operate with a maximum amplitude of 0.5 m at frequencies well below resonance. The engineer also wants the resonant frequency to be equal to 1 Hz and the machine to respond well to all frequencies below 1 Hz. The block weighs 1000 kg. What values of s and r should the engineer use? Calculate the maximum displacement at resonance.

[3 marks]

[TOTAL 18 marks]