Vibrations & Waves Homework Sheet 1
ANSWERS

1) (i) $x(t) = 2\exp(j6t) = 2\left[\cos(6t) + j\sin(6t)\right]$

=> Real part: $x(t) = 2\cos(6t)$

(ii) $x(t) = j3\exp(j6t) = 3\left[j\cos(6t) + (-1)\sin(6t)\right]$

=> Real part: $x(t) = -3\sin(6t)$

or $x(t) = j3\exp(j6t) = 3\exp(j(6t+\phi))$ where $\phi = \arctan(3/0) = \pi/2$

$x(t) = 3\exp(j(6t+\pi/2)) = 3\left[\cos(6t+\pi/2) + j\sin(6t+\pi/2)\right]$

=> Real part: $x(t) = 3\cos(6t+\pi/2) = -3\sin(6t)$

(iii) $x(t) = (2+j3)\exp(j6t) = A\exp(j(6t+\phi))$

where $\phi = \arctan(3/2) = 0.983$ and $A = \sqrt{2^2+3^2} = 3.606$

$x(t) = 3.606\exp(j(6t+0.983))$

$= 3.606\left[\cos(6t+0.983) + j\sin(6t+0.983)\right]$

=> Real part: $x(t) = 3.606\cos(6t+0.983)$

(iv) $x(t) = (2-j5)\exp(j6t) = A\exp(j(6t+\phi))$

where $\phi = \arctan(-5/2) = -1.190$ and $A = \sqrt{2^2+5^2} = 5.385$

$x(t) = 5.385\exp(j(6t-1.190))$

$= 5.385\left[\cos(6t-1.190) + j\sin(6t-1.190)\right]$

=> Real part: $x(t) = 5.385\cos(6t-1.190)$

2)

(i) $x(t) = 5\cos(8t) = 5[\cos(8t) + j\sin(8t)] = 5\exp(j8t)$

(ii)

$$
\begin{aligned}
x(t) &= 5\cos(8t + 0.3\pi) \\
&= 5[\cos(8t + 0.3\pi) + j\sin(8t + 0.3\pi)] \\
&= 5\exp(j(8t + 0.3\pi)) = 5\exp(j0.3\pi)\exp(j8t) \\
&= 5[\cos(0.3\pi) + j\sin(0.3\pi)]\exp(j8t) \\
&= [1.763 + j2.427]\exp(j8t)
\end{aligned}
$$

(iii)

$$
\begin{aligned}
x(t) &= 7\cos(5t - 0.2\pi) \\
&= 7[\cos(5t - 0.2\pi) + j\sin(5t - 0.2\pi)] \\
&= 7\exp(j(5t - 0.2\pi)) = 7\exp(-j0.2\pi)\exp(j5t) \\
&= 7[\cos(-0.2\pi) + j\sin(-0.2\pi)]\exp(j5t) \\
&= 7[\cos(0.2\pi) - j\sin(0.2\pi)]\exp(j5t) \\
&= [5.663 - j4.114]\exp(j5t)
\end{aligned}
$$

(iv)

$$
\begin{aligned}
x(t) &= 8\sin(7t) = 8\cos(7t - \pi/2) \\
&= 8[\cos(7t - \pi/2) + j\sin(7t - \pi/2)] \\
&= 8\exp(j(7t - \pi/2)) = 8\exp(-j\pi/2)\exp(j7t) \\
&= 8[\cos(-\pi/2) + j\sin(-\pi/2)]\exp(j7t) \\
&= -j8\exp(j7t)
\end{aligned}
$$

3) (i) $\boldsymbol{A}$ = 0.07m

(ii) ω_0 = 5.71 rads/s

(iii) $\boldsymbol{f}$ = 0.909 Hz

(iv) $\boldsymbol{T}$ = 1.10 s

$\omega_0 = \sqrt{s/m}$ => $s = m\omega_0^2$ Hence $\boldsymbol{s}$ = 3.26 N/m

Assume the spring stretches a distance **-*L*** downwards (in negative x direction) to balance out force due to gravity **-*mg*** (also in negative x direction).

Restoring force (in positive x direction): $F = -sx = sL$

No acceleration => total force: $sL - mg = 0$

Therefore: $L = mg/s$

=> $\boldsymbol{L}$ = 0.301 m.

4) $x(t) = A\cos(5t + \phi)$

=> $v(t) = -5A\sin(5t + \phi)$

By considering the initial conditions, work out the value of A and ϕ for the following cases:

(i) t = 0, x = 0.3 m, v = 0

=> $x(t=0) = 0.3$

=> A = 0.3 m and ϕ = 0

(i) t = 0, x = -0.5 m, v = 0

=> $x(t=0) = -0.5$

=> A = -0.5 m and ϕ = 0 (or A = 0.5 m and ϕ = π)

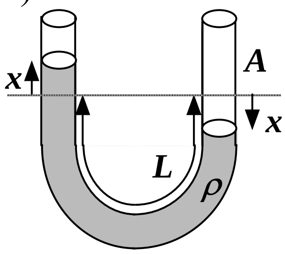
(iii) t = 0, x = 0, v = 1.2 m/s

=> $x(t=0) = A\cos(0 + \phi) = 0$ => ϕ = π/2

. $v(t=0) = -5A\sin(0 + \pi/2) = -5A = 1.2$

=> A = -1.2/5 = -0.24 m

5)



(i) Define $\boldsymbol{x}$ as displacement upwards on left hand side of tube.
Restoring force due to displacement of liquid upwards on the left hand side of tube: $F_{left} = -A\rho g x$
Restoring force due to displacement of liquid downwards on the right hand side of tube *in **x** direction*: $F_{right} = -A\rho g x$
Therefore total restoring force: $F = -2A\rho g x$
Total mass of liquid: $m = LA\rho$
Equation of motion: $LA\rho\dfrac{d^2x}{dt^2} = -2A\rho g x$
(ii)
$$\boldsymbol{LA\rho\frac{d^2x}{dt^2} = -2A\rho g x} \quad => \quad \boldsymbol{\frac{d^2x}{dt^2} = -\frac{2g}{L}x}$$
Trial solution: $\boldsymbol{x(t) = \underline{A}\exp(j\omega_0 t)}$
$$=> \boldsymbol{\frac{d^2x}{dt^2} = -\omega_0^2\underline{A}\exp(j\omega_0 t) = -\frac{2g}{L}\underline{A}\exp(j\omega_0 t)}$$
Trial solution is valid.
Real part: $\boldsymbol{x(t) = A\cos(\omega_0 t + \phi)}$
At $t = 0$, $\boldsymbol{x} = \boldsymbol{h}$, $v = 0$: Therefore: $\boldsymbol{x(t) = h\cos(\omega_0 t)}$

(iii) $\boldsymbol{\omega_0 = \sqrt{2g/L}}$

(iv) $v(t) = \dfrac{dx(t)}{dt} = -h\omega_0\sin(\omega_0 t)$

(v) $a(t) = \dfrac{dv(t)}{dt} = -h\omega_0^2\cos(t)$

(vi) $PE = -\int F.dx = -\int -2A\rho g x dx$
$PE = A\rho g x^2 + U_0$ where $\boldsymbol{U_0}$ is a constant.
When $\boldsymbol{x} = 0$, $\boldsymbol{PE} = 0$. Therefore $\boldsymbol{U_0} = 0$.
Therefore: $PE = A\rho g x^2$
PE in terms of time: $PE = A\rho g(h\cos(\omega_0 t))^2 = A\rho g h^2(\cos(\omega_0 t))^2$

(vii) $KE = \frac{1}{2}mv^2 = \frac{1}{2}LA\rho\left(-h\omega_0 \sin(\omega_0 t)\right)^2$

$=> KE = \frac{1}{2}LA\rho h^2 \omega_0^{\;2} \left(\sin(\omega_0 t)\right)^2$

$\omega_0 = \sqrt{2g/L} => L\omega_0^{\;2} = 2g$

$=> KE = A\rho g h^2 \left(\sin(\omega_0 t)\right)^2$

(viii) Total energy: $TE = PE + KE = A\rho g h^2$

(ix) $KE = TE - PE = A\rho g(h^2 - x^2)$

6)i)

General complex solution:

$$\boldsymbol{x = A_1 \exp(j\omega_0 t) + A_2 \exp(j2\omega_0 t) + A_3 \exp(j3\omega_0 t) + ...}$$

General real solution of x(t):

$$\boldsymbol{x = A_1 \cos(\omega_0 t) + A_2 \cos(2\omega_0 t) + A_3 \cos(3\omega_0 t) + ...}$$

ii)

We would expect:

$A_1 > A_2 > A_3$

iii)

Equation of Motion

$$\boldsymbol{m\frac{d^2x}{dt^2} = -50x - 3x^2 - 0.06x^3}$$

Trial solution:

$$\boldsymbol{x = A_1 \exp(j\omega_0 t) + A_2 \exp(j2\omega_0 t) + A_3 \exp(j3\omega_0 t) + ...}$$

Solve:

$$\boldsymbol{-m\omega_0^2 A_1 \exp(j\omega_0 t) - m4\omega_0^2 A_2 \exp(j2\omega_0 t)}$$
$$\boldsymbol{-m9\omega_0^2 A_3 \exp(j3\omega_0 t)....}$$
$$\boldsymbol{= -50\left[A_1 \exp(j\omega_0 t) + A_2 \exp(j2\omega_0 t) + A_3 \exp(j3\omega_0 t) + ...\right]}$$
$$\boldsymbol{-3\left[A_1^2 \exp(j2\omega_0 t) + 2A_1 A_2 \exp(j3\omega_0 t) + ...\right]}$$
$$\boldsymbol{-0.06\left[A_1^3 \exp(j3\omega_0 t) + ...\right]}$$

[NOTE: this is the corrected version – original left out factor of 2 in front of A_1A_2 term]

Take terms of similar order from both sides:

1st Harmonic:

$$-m\omega_0^2 A_1 \exp(j\omega_0 t) = -50A_1 \exp(j\omega_0 t)$$

$$=> \omega_0^2 = \frac{50}{m}$$

2nd Harmonic:

$$-m4\omega_0^2 A_2 \exp(j2\omega_0 t) = -50A_2 \exp(j2\omega_0 t) - 3A_1^2 \exp(j2\omega_0 t)$$

$$=> 4\omega_0^2 A_2 = \frac{50}{m} A_2 + \frac{3}{m} A_1^2$$

$$=> 4\omega_0^2 - \frac{50}{m} = \frac{3}{m}\frac{A_1^2}{A_2}$$

$$=> 3\frac{50}{m} = \frac{3}{m}\frac{A_1^2}{A_2}$$

$$=> A_2 = \frac{A_1^2}{50}$$

$$=> A_2 = 0.02A_1^2$$

3rd Harmonic:

$$m9\omega_0^2 A_3 = 50A_3 + 6A_1A_2 + 0.06A_1^3$$

$$=> 9\frac{50}{m} A_3 = \frac{50}{m} A_3 + \frac{6}{m} A_1A_2 + \frac{0.06}{m} A_1^3$$

$$=> 8\frac{50}{m} A_3 = \frac{6}{m} A_1A_2 + \frac{0.06}{m} A_1^3$$

$$=> A_3 = \frac{6A_1A_2 + 0.06A_1^3}{8 \times 50}$$

$$=> A_3 = \frac{(6/50)A_1^3 + 0.06A_1^3}{8 \times 50}$$

$$=> A_3 = \frac{[0.12 + 0.06]A_1^3}{8 \times 50}$$

$$A_3 = 0.00045A_1^3$$

[NOTE: this is the corrected version]