SoM: Problem Sheet 2, Answers

1. $$\int_{v_x=-\infty}^{v_x=+\infty} f(v_x)\,dv_x = 1$$

$$\text{LHS} = \int_{-\infty}^{+\infty} A e^{-\alpha v_x^2}\,dv_x = A\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} \text{ (standard integral)}$$

$$\rightarrow A = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}} = \left(\frac{m}{2\pi k_B T}\right)^{\frac{1}{2}}$$

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2. v_{mp} corresponds to peak in $f(v)$.

$$\frac{df}{dv} = 4\pi A^3 \frac{d}{dv}\left(v^2 e^{-\alpha v^2}\right) = 4\pi A^3\left(2v e^{-\alpha v^2} - v^2 2\alpha v e^{-\alpha v^2}\right)$$

$$= 0 \text{ at max} \rightarrow 1 - \alpha v^2 = 0$$

$$\therefore v = \left(\frac{1}{\alpha}\right)^{\frac{1}{2}} = \left(\frac{2k_B T}{m}\right)^{\frac{1}{2}}$$

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3. Using standard integral

$$\langle v \rangle = 4\pi A^3 \times \frac{1}{2\alpha^2} = 2\pi \times \left(\frac{m}{2\pi k_B T}\right)^{3/2} \times \left(\frac{2k_B T}{m}\right)^2$$

$$= \left(\frac{8k_B T}{\pi m}\right)^{\frac{1}{2}}$$

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4. $$\langle v^2 \rangle = \int_0^{\infty} v^2 f(v)\,dv = 4\pi A^3 \int_0^{\infty} v^4 e^{-\alpha v^2}\,dv$$

$$= 4\pi A^3 \times \frac{3\pi^{1/2}}{8\alpha^{5/2}} \quad \text{(using standard integral)}$$

$$= \frac{3}{2}\pi^{3/2}\left(\frac{m}{2\pi k_B T}\right)^{3/2} \times \left(\frac{2k_B T}{m}\right)^{5/2} = \frac{3k_B T}{m}$$

$$\therefore \langle ke \rangle = \frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} k_B T \quad \text{(cf sec 7.3)}$$

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5. m = mass of O_2 molecule

$$= 2 \times 16.0 \times 1.66 \times 10^{-27} = 5.31 \times 10^{-26}\ \text{kg}$$

$$\left(\frac{k_B T}{m}\right)^{1/2} = \left(\frac{1.38 \times 10^{-23} \times 293}{5.31 \times 10^{-26}}\right)^{1/2} = 276\ \text{ms}^{-1}$$

$$v_{mp} = \sqrt{2}\left(\frac{k_B T}{m}\right)^{1/2} = 390\ \text{ms}^{-1}$$

$$\langle v \rangle = \sqrt{\frac{8}{\pi}}\left(\frac{k_B T}{m}\right)^{1/2} = 440\ \text{ms}^{-1}$$

$$v_{rms} = \sqrt{3}\left(\frac{k_B T}{m}\right)^{1/2} = 478\ \text{ms}^{-1}$$

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6 (a) $$\langle v \rangle = \left(\frac{8k_B T}{\pi m}\right)^{1/2} = \left(\frac{8 \times 1.38 \times 10^{-23} \times 10^8}{\pi}\right)^{1/2} \times \frac{1}{\sqrt{m}}$$

$$= \frac{5.93 \times 10^{-8}}{\sqrt{m}}$$

$$\langle v \rangle_D = 1.03 \times 10^6\ \text{ms}^{-1}$$

$\langle v \rangle_T = 8.36 \times 10^5 \ ms^{-1}$

$\langle v \rangle_e = 6.21 \times 10^7 \ ms^{-1}$

(b) $\langle E \rangle = \frac{3}{2} k_B T = 2.07 \times 10^{-15}$ J for all 3 types

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7. $\langle v_x \rangle = \int_{-\infty}^{+\infty} v_x f(v_x) dv_x = A \underbrace{\int_{-\infty}^{+\infty} v_x e^{-\alpha v_x^2} dv_x}_{= 0 \text{ (standard integral)}}$

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8 $\langle z \rangle = \int_0^\infty z \, p(z) dz = \int_0^\infty \frac{z}{\lambda} e^{-z/\lambda} dz$

make subs $\xi = z/\lambda$, $dz = \lambda d\xi$

$\langle z \rangle = \lambda \int_0^\infty \xi e^{-\xi} d\xi$

$= \lambda \underbrace{\left[-\xi e^{-\xi} \right]_0^\infty}_{= 0} + \lambda \int_0^\infty e^{-\xi} d\xi$

$= \lambda \left[-e^{-\xi} \right]_0^\infty = -\lambda(0-1) = \lambda = \frac{k_B T}{mg}$

$\langle pe \rangle = mg \langle z \rangle = k_B T$
