Structure of Matter
First Year Physics

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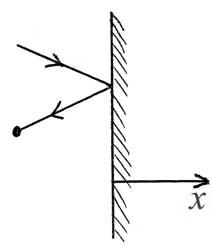
Classwork III
Derivation of the Ideal Gas Equation of State

In Section 1.1 of Lecture 1 we derived the ideal gas equation of state by considering gas molecules undergoing elastic collisions with the wall of the container and imparting momentum to it. The pressure is just the rate of momentum transfer per unit area. However, one aspect of that derivation was highly unsatisfactory. In this Classwork we do it properly.

Consider a box of volume V containing N molecules. Define the x axis pointing into the wall, as shown. The dodgy part of the previous derivation was the assumption that all the molecules had the same value of $|v_x|$, and then at the end replacing $|v_x|^2$ with the average value, without any real justification. In fact, of course, the molecules have values of v_x between $-\infty$ and $+\infty$, according to the velocity component distribution function: $f(v_x) = A\mathrm{e}^{-\alpha v_x^2}$ where $A = \left(\dfrac{m}{2\pi k_B T}\right)^{1/2}$ and $\alpha = \dfrac{m}{2k_B T}$.



1. Write down an expression for the total number of molecules in the volume V with v_x between v_x and $v_x + \mathrm{d}v_x$, and, hence, show that the number density of such molecules (the number per unit volume) is $nf(v_x)\mathrm{d}v_x$ where $n = N/V$.

2. We need to consider the range $0 < v_x < +\infty$ (molecules with $v_x < 0$ are moving away from the wall and won't hit it). Show that the number of impacts with area L^2 of the wall[1] in time Δt by molecules with v_x between v_x and $v_x + \mathrm{d}v_x$ is $N_{coll} = L^2 v_x n f(v_x)\mathrm{d}v_x \Delta t$.

3. In Sec. 1.1 we found (replacing the A which was used there by L^2) $N_{coll} = \dfrac{1}{2}L^2|v_x|n\Delta t$. Why doesn't the factor $\dfrac{1}{2}$ appear in the new equation for N_{coll}, in Q. 2?

4. Each impact imparts momentum $2mv_x$ to the wall (remember, we only consider positive values of v_x). Show that the contribution to the pressure on the wall from the molecules between v_x and $v_x + \mathrm{d}v_x$ is $\mathrm{d}P = 2nmv_x^2 f(v_x)\mathrm{d}v_x$.

5. Using the standard integral $\displaystyle\int_0^\infty x^2 \mathrm{e}^{-\alpha x^2}\mathrm{d}x = \frac{1}{4}\left(\frac{\pi}{\alpha^3}\right)^{1/2}$ show that $P = nk_B T$ (the ideal gas equation of state).

[1] In Sec. 1.1 we considered an area A of the wall, but here we are using the symbol A for the normalizing constant in $f(v_x)$.