Vibrations and Waves
Problem Sheet 1: Answers

1. Note: to be consistent with notation in lectures, a "~" has been put over the complex functions.
   - i. $\tilde{x}(t) = 2\exp(i6t) = 2[\cos(6t) + i\sin(6t)] \Rightarrow \text{Re}[\tilde{x}(t)] = 2\cos(6t)$
   - ii. $\tilde{x}(t) = i\,3\exp(i5t) = 3[i\cos(5t) - \sin(5t)] \Rightarrow \text{Re}[\tilde{x}(t)] = -3\sin(5t)$
   - iii. $\tilde{x}(t) = (2 + i3)\exp(i6t) = (2 + i3)[\cos(6t) + i\sin(6t)] = 2\cos(6t) - 3\sin(6t) + i[3\cos(6t) + 2\sin(6t)] \Rightarrow \text{Re}[\tilde{x}(t)] = 2\cos(6t) - 3\sin(6t)$. This is <u>not</u> in a very useful form to see what's going on. Better if we can get the answer in the form $A\cos(\omega t + \varphi)$. This can be done by writing $(2 + 3i)$ in our original $\tilde{x}(t)$ in the form $R\exp(i\theta)$, where $R = \sqrt{2^2 + 3^2} = 3.61$ and $\theta = \arctan(3/2) = 56.3° = 0.983$rad. So $\tilde{x}(t) = 3.61\exp(i\,0.983)\exp(i\,6t) = 3.61\exp[i(6t + 0.983)] = 3.61[\cos(6t + 0.983) + i\sin(6t + 0.983)] \Rightarrow \text{Re}[\tilde{x}(t)] = 3.61\cos(6t + 0.983)$.
   - iv. $R\tilde{x}(t) = (1 - 5i)\exp(i2t) = \sqrt{1^2 + (-5)^2}\exp[i\arctan(-5)]\exp(i2t) = 5.10\exp(-i\,1.37)\exp(i2t) = 5.10\exp[i(2t - 1.37)] = 5.10[\cos(2t - 1.37) + i\sin(2t - 1.37)] \Rightarrow \text{Re}[\tilde{x}(t)] = 5.10\cos(2t - 1.37)$.

2.
   - i. $x(t) = 5\cos(8t) \Rightarrow \tilde{x}(t) = (5 + i\,0)\exp(i8t)$ – check: $\text{Re}[\tilde{x}(t)] = 5\cos(8t)$.
   - ii. $x(t) = 5\cos(8t + 0.2\pi) \Rightarrow \tilde{x}(t) = 5\exp[i(8t + 0.2\pi)] = 5\exp(i\,0.2\pi)\exp(i8t) = 5[\cos(0.2\pi) + i\sin(0.2\pi)]\exp(i8t) = (4.05 + i\,2.94)\exp(i8t)$.
   - iii. $x(t) = 7\cos(5t - 0.3\pi) \Rightarrow \tilde{x}(t) = 7\exp[i(5t - 0.3\pi)] = 7\exp(-0.3\pi)\exp(i5t) = 7[\cos(-0.3\pi) + i\sin(-0.3\pi)]\exp(i5t) = (4.11 - i\,5.66)\exp(i5t)$.
   - iv. $x(t) = 5\sin(7t) = 5\cos(\pi/2 - 7t) = 5\cos(7t - \pi/2) \Rightarrow \tilde{x}(t) = 5\exp[i(7t - \pi/2)] = 5\exp(-i\pi/2)\exp(i7t) = 5[\cos(-\pi/2) + i\sin(-\pi/2)]\exp(i7t) = (0 - i\,5)\exp(i7t)$.

3. $x(t) = 0.05\cos(7.51\,t)$
   - i. amplitude A = 0.05 m
   - ii. $\omega = 7.51$ rad/s = 7.51 /s (although rad is dimensionless, keeping the rad in the units reminds us that its an angular frequency).
   - iii. $f = \omega/(2\pi) = 1.20$ Hz
   - iv. $T = 1/f = 0.84$ s

     $\omega^2 = s/m \Rightarrow s = m\omega^2 = 0.1\text{ kg }(7.51\text{ rad/s})^2 = 5.64\text{ kg/s}^2 = 5.64$ N/m

     Assume spring stretches a distance L downwards (in positive x-direction). The restoring force due to spring is –sL (in negative x-direction, i.e. up). Spring will stretch until this restoring force balances force of gravity mg (in positive x-direction), i.e. $mg - sL = 0 \Rightarrow L = mg/s = (0.1 \times 9.8)\text{ N}/5.64\text{ N/m} = 0.17$ m

4. $x(t) = A\cos(4t + \varphi)$, $v(t) = dx/dt = -4A\sin(4t + \varphi)$
   - i. $x(0) = 0.3\text{ m} \Rightarrow A\cos\varphi = 0.3$ m; $v(0) = 0 \Rightarrow -4A\sin\varphi = 0 \Rightarrow \boxed{\varphi = 0}$, so $A\cos 0 = 0.3\text{ m} \Rightarrow \boxed{A = 0.3\text{ m}}$.
   - ii. $x(0) = -0.5\text{m} \Rightarrow A\cos\varphi = -0.5$ m; $v(0) = 0 \Rightarrow -4A\sin\varphi = 0 \Rightarrow \boxed{\varphi = 0}$ so $A\cos 0 = -0.5\text{m} \Rightarrow \boxed{A = -0.5\text{m}}$
   - iii. $x(0) = 0 \Rightarrow A\cos\varphi = 0 \Rightarrow \boxed{\varphi = \pi/2}$; $v(0) = 1.2\text{ m/s} \Rightarrow -4A\sin(\pi/2) = 1.2\text{ m/s} \Rightarrow \boxed{A = -0.3\text{ m.}}$

5.
   - i. When the liquid in the left and right hand sides of the tube is not at the same height, there is a force on the liquid due to the weight of the displaced liquid. If the height of the liquid on the left hand side goes up by x, the height on the right hand side must go down by x (assuming a constant cross section of the tube and an incompressible liquid), so the difference in the heights is $2x$. The mass of this amount of liquid is m = area × height × density = $A2x\rho$, so the weight is $F = -mg = -(2A\rho g)x$ (minus sign ⇒ down). This provides

a restoring force proportional to x and in the opposite direction to the motion ($\Rightarrow$ Hooke's Law so we expect SHM). By Newton II, Ma = restoring force, where a = d^2x/dt^2 is the acceleration of the liquid and $M = AL\rho$ is the total mass of liquid in the tube. Therefore $AL\rho\frac{d^2x}{dt^2} = -2A\rho g x$, or $\frac{d^2x}{dt^2} = -\frac{2g}{L}x$ as required.

ii. We know that the general solution for this kind of equation is $x(t) = A\cos(\omega t + \varphi)$. Let's use the complex form $\tilde{x}(t) = A\exp[i(\omega t + \varphi)]$ (remembering that the actual displacement $x(t)$ is just the real part of this) to check this, and to derive an expression for ω. $\frac{d\tilde{x}}{dt} = i\omega A\exp[i(\omega t + \varphi)]$, $\frac{d^2\tilde{x}}{dt^2} = -\omega^2 A\exp[i(\omega t + \varphi)]$ Substitute $\tilde{x}$, $\frac{d^2\tilde{x}}{dt^2}$ into equation of motion to get: $\omega^2 = 2g/L$, so $\tilde{x}(t) = A\exp[i(\omega t + \varphi)]$ <u>is</u> a solution provided $\omega = \sqrt{2g/L}$. The initial conditions are $x(0) = h \Rightarrow A\cos\varphi = h$; and $v(0) = 0 \Rightarrow -A\omega\sin\varphi = 0 \Rightarrow \varphi = 0$. Therefore, $A\cos 0 = h \Rightarrow A = h$. Therefore exact solution for this situation is $x(t) = \mathrm{Re}[\tilde{x}(t)] = h\cos\left[\left(\sqrt{2g/L}\right)t\right]$

iii. Liquid oscillates at angular frequency $\omega = \sqrt{2g/L}$.

iv. $v(t) = -h\omega\sin(\omega t)$

v. $a(t) = -h\omega^2\cos(\omega t)$

vi. Work done displacing liquid from position x to x+dx is $dW = -Fdx = 2A\rho g x dx$. PE is total work going from 0 to x: $PE = \int_0^x dW = 2A\rho g\int_0^x x dx = A\rho g x^2$. So PE as a function of time is $PE = A\rho g h^2\cos^2(\omega t)$

vii. $KE = \frac{1}{2}Mv^2 = \frac{1}{2}AL\rho h^2\omega^2\sin^2(\omega t)$. But $\omega = \sqrt{2g/L} \Rightarrow L\omega^2 = 2g \Rightarrow KE = A\rho g h^2\cos^2(\omega t)$

viii. Total energy $E = PE + KE = A\rho g h^2$ which is constant, as expected.

ix. $KE(x) = E - PE(x) = A\rho g\left(h^2 - x^2\right)$

6. Taylor series expand $U(x)$ about x_0:

$$U(x) = U(x_0) + U'(x_0)(x - x_0) + \tfrac{1}{2!}U''(x_0)(x - x_0)^2 + \tfrac{1}{3!}U'''(x_0)(x - x_0)^3 + \ldots$$

Because x_0 is a stable equilibrium, $U'(x_0) = 0$ and $U''(x_0) > 0$ (local minimum – see fig. below). Therefore the force $F(x) = -dU(x)/dx$ is given by $F(x) = -U''(x_0)(x - x_0) - \tfrac{1}{2}U'''(x_0)(x - x_0)^2 - \ldots$ since $U(x_0)$ = const. For sufficiently small displacements, $(x - x_0)$, we can neglect the second term proportional to $(x - x_0)^2$ and higher order terms compared to $(x - x_0)$, and because $U''(x_0) > 0$ we can write $F(x) = -s(x - x_0)$ where $s > 0$. Hence there is a restoring force (minus sign) that is linear in the displacement from equilibrium. Note: in the lectures we set the equilibrium position to be at $x = 0$ for convenience, which leads to the familiar form of Hooke's Law $F(x) = -sx$.