Structure of Matter

Problem Sheet 4 **Answers**

1. (a) Iceberg is in equilibrium so
   weight of berg = buoyant force on cube = weight of water displaced

   If volume displaced = V_1 and total volume = V

   $\rightarrow \rho_{ice}\ V\ g = \rho_{water}\ V_1\ g$

   hence fraction submerged = $V_1/V = \rho_{ice}/\rho_{water} = 920/1025 = 0.898$ (89.8%)

   (b) For balloon to be in equilibrium,
   weight of balloon + weight of helium filled = weight of air displaced

   $5 \times 10^{-3}\, g + \frac{4}{3}\pi r^3 * \rho_{He}\, g = \frac{4}{3}\pi r^3 * \rho_{air}\, g$

   $5 \times 10^{-3} = \frac{4}{3}\pi r^3\ n_0\ (m_{air} - m_{He})$

   $$r^3 = \frac{5 \times 10^{-3}}{\frac{4}{3}\pi\ (1.01 \times 10^5/k_B * 273)\ (29 - 4)\ 1.66 \times 10^{-27}}$$

   $r = 0.10$ m $= 10$ cm

2. (a) Incompressible so $u_1 A_1 = u_2 A_2$

   $u_2 = \pi 0.8^2 * 1.2/\pi 0.4^2 = 4.8$ m/s

   (b) Flow rate $= u.A = \pi(0.4 \times 10^{-2})^2 * 4.8 = 2.41 \times 10^{-4}$ m^3 /s

   So time taken to fill tank $= 20/2.41 \times 10^{-4} = 8.29 \times 10^4$ s ≈ 23 hours

3. (a) Effective volume is $b = 4V_m = 4\frac{4}{3}\pi r^3$

   so $r = (3b/16\pi)^{1/3} = (3 \cdot 6.49 \times 10^{-29}/16\pi)^{1/3} = 1.57 \times 10^{-10}$m $= 1.57$ Å

   (b) $$T_c = \frac{8a}{27 k_B b} = \frac{8 \cdot 3.86 \times 10^{-49}}{27 k_B 6.49 \times 10^{-29}} = 128 \text{ K}$$

4. (a) 3 directions of vibration with 2 degrees of freedom for each direction (one pe and one ke) $\rightarrow n_d = 6$

$U = 6 \times \frac{1}{2} N k_B T = 3 N k_B T$,

$$C_V = \left(\frac{\partial Q}{\partial T}\right)_V = \left(\frac{\mathrm{d}U}{\mathrm{d}T}\right) \text{ (from first law with no work done due to constant V)}$$

$\rightarrow C_V = 3 N k_B$

(b) For 1 mole $C_{V_m} = 3 N_A k_B = 3R = 24.9$ J/K per mole $= 5.96$ cal/K

5. (a) $\Delta L = 23.1 \times 0.3 \times (77 - 293)\ \mu\text{m} = -1.5 \times 10^3\ \mu\text{m} = -1.5$ mm

(a) $\Delta L = 1.17 \times 10^{-5} \times 320 \times (-30) = -0.112\,\text{m} \sim -11$ cm

(b) Volume $V = L^3 = \{L_0(1 + \alpha \Delta T)\}^3 = L_0^3 (1 + \alpha \Delta T)^3$

$\approx L_0^3 (1 + 3\alpha \Delta T + ...)$ ignoring terms in α^2 or higher due to smallness

Since $V_0 = L_0^3$, then $\Delta V = V_0\, 3\alpha \Delta T = V_0\, \beta \Delta T$, i.e. $\beta = 3\alpha$

6. (a) State zero (ground) has probability $p_0 = A\mathrm{e}^{-E/k_B T}$
State one has probability $p_1 = A\mathrm{e}^{-(E+\Delta E)/k_B T}$

Total probability = 1,

so $A\mathrm{e}^{-E/k_B T} + A\mathrm{e}^{-(E+\Delta E)/k_B T} = 1$

$\rightarrow \quad A = 1/\{\mathrm{e}^{-E/k_B T}(1 + \mathrm{e}^{-\Delta E/k_B T})\}$

So $N_0 = N p_0 = \dfrac{N}{(1 + \mathrm{e}^{-\Delta E/k_B T})}$

and $N_1 = N p_1 = \dfrac{N\mathrm{e}^{-\Delta E/k_B T}}{(1 + \mathrm{e}^{-\Delta E/k_B T})}$

(b) Relative proportions $N_1/N_0 = \mathrm{e}^{-\Delta E/k_B T}$, so

i. $k_B T = 0.1\,\Delta E \quad \textit{ratio} = \mathrm{e}^{-0.1} = 4.5 \times 10^{-5}$

ii. $k_B T = 0.3\,\Delta E \quad \textit{ratio} = 3.6 \times 10^{-2}$

iii. $k_B T = \Delta E \quad \textit{ratio} = 0.368$

iv. $k_B T = 10\,\Delta E \quad \textit{ratio} = 0.905$

(c) $E = 0.023 * 1.602 \times 10^{-19} = 3.68 \times 10^{-21} J = k_B T$

$T = 267$ K. This compares with its melting point which is 312 K

(d) $E = 4.2 * 1.602 \times 10^{-19} = 6.73 \times 10^{-19} J = k_B T$

$T = 48,800$ K. This corresponds to conversion to a plasma

(e) $\lambda = h/p > n^{-1/3}$, but $\langle p \rangle^2/2m = \frac{3}{2}k_B T$

$\rightarrow \quad \langle p \rangle = (3k_B T m)^{1/2}$

$\rightarrow \quad h/(3k_B T m)^{1/2} > n^{-1/3}$

$\rightarrow \quad T < (h^2/3mk_B)\, n^{2/3}$

(f) $n = 2000/(1 \times 10^{-5})^3 = 2 \times 10^{18}$ m^{-3}

So $T < ((6.626 \times 10^{-34})^2/(3 \cdot 86 \cdot 1.66 \times 10^{-27} \cdot 1.38 \times 10^{-23}))\ (2 \times 10^{18})^{2/3}$

$T < 1.18 \times 10^{-7}$ K $= 118$ nK

Corresponding energy for this temperature $= k_B T = 1.18 \times 10^{-7} \cdot 1.38 \times 10^{-23} = 1.63 \times 10^{-30}$ J $= 10^{-11}$ eV

7. (a) For adiabatic change $PV^\gamma = A$ (= constant)

$$P = AV^{-\gamma} \quad \rightarrow \quad \frac{\mathrm{d}P}{\mathrm{d}V} = -A\gamma V^{-\gamma-1} = -\gamma P/V$$

So $K = \gamma P$ and $c_s = \sqrt{K/\rho} = \sqrt{\gamma P/\rho}$

(b) But for ideal gas $\rho = n_0 \times m = \dfrac{mP}{k_B T}$

so $c_s = \sqrt{\gamma k_B T/m}$

For air, $m = 29 * 1.66 \times 10^{-27}$, $\gamma = 1.4$, $c_s = \sqrt{\dfrac{1.4 * k_B * 273}{29 * 1.66 \times 10^{-27}}} = 331$ m/s

For He, $m = 4 * 1.66 \times 10^{-27}$, $\gamma = 5/3$, $c_s = \sqrt{\dfrac{(5/3) * k_B * 273}{4 * 1.66 \times 10^{-27}}} = 973$ m/s

8. (a) The E field can be obtained from Gauss's law, consider a cylinder, with one flat face (of area A) in the plane between the separated charges, and the other far from the charges where the E field is zero. By symmetry there is no field through the curved sides, so integrating over the side which has only ions in a layer of

thickness δ

$$E \cdot A = q/\epsilon_0 = n_0 \cdot e \cdot \delta \cdot A/\epsilon_0$$

$$\rightarrow \quad E = n_0 e\delta/\epsilon_0$$

(b) $m_e \dfrac{\mathrm{d}\delta}{\mathrm{d}t} = -eE_{max} = -en_0 e\delta/\epsilon_0$

$$\rightarrow \delta'' + \left(\frac{n_0 e^2}{m_e \epsilon_0}\right) \delta = 0$$

which is SHM with angular frequency $\omega_p = \left(\dfrac{n_0 e^2}{m_e \epsilon_0}\right)^{1/2}$ - the plasma frequency

(d) $u_E = \frac{1}{2}\epsilon_0 \, E^2 = \frac{1}{2}\epsilon_0 \, (n_0 e\delta/\epsilon_0)^2 = \frac{1}{2}(n_0^2 e^2 \delta^2/\epsilon_0)$

(e) $\frac{1}{2} n_0 k_B T = \frac{1}{2}(n_0^2 e^2 \delta^2/\epsilon_0)$

rearrange to get, $\delta_{max} = \left(\dfrac{\epsilon_0 k_B T}{n_0 e^2}\right)^{1/2}$

(f) $\omega_p = \left(\dfrac{n_0 e^2}{m_e \epsilon_0}\right)^{1/2} = \left(\dfrac{10^{20} e^2}{9.1 \times 10^{-31} \; 8.85 \times 10^{-12}}\right)^{1/2} = 5.6 \times 10^{11}\text{Hz}$

$$\delta_{max} = \left(\frac{\epsilon_0 k_B T}{n_0 e^2}\right)^{1/2} = \left(\frac{8.85 \times 10^{-12} \cdot 1.38 \times 10^{-23} \cdot 10^8}{10^{20} \cdot (1.60 \times 10^{-19})^2}\right)^{1/2} = 6.91 \times 10^{-5} \text{ m}$$