Structure of Matter 21.05.08
First Year Physics

Problem Sheet 4
Lectures 9-11

Learning Outcomes

Jargon

Buoyancy, mass flow rate, isobar, isochor, critical point, triple point, melting, vaporization, sublimation, metallic bond, linear and volume thermal expansion coefficients, sound waves, bulk modulus Bose-Einstein condensate, ionisation, plasma.

Concepts

Pressure gradient force in fluid; Archimedes' Principle; continuity equation; classical expressions for the internal energy and constant volume heat capacity of a solid; thermal expansion; calculating the frequency of sound waves, calculating plasma frequency and Debye length.

Problems

1. (a) An iceberg of density 920 kg m^{-3} floats in seawater of density 1025 kg m^{-3}. What fraction of its volume is submerged?

   (b) To what radius must an approximately spherical balloon of mass 5g be filled with helium gas for it be able to support its own weight due to the buoyant force of the displaced air? (Assume both air and He are at STP).

2. (a) Water flows at 1.2 m s^{-1} through a hose-pipe of radius 0.8 cm. Calculate the speed with which it emerges from a nozzle of radius 0.4 cm.

   (b) How long would it take to fill a tank of volume 20 m^3 with this hose-pipe?

3. The van der Waals constants for Nitrogen are $a = 3.86 \times 10^{-49}$ J m^3 and $b = 6.49 \times 10^{-29}$ m^3.

   (a) Assuming that a Nitrogen molecule is spherical, estimate its radius.

   (b) Use the result of Q8, Problem Sheet 3, to estimate the critical temperature of Nitrogen. (For comparison the actual value is 126 K.)

4. (a) Using the law of equipartition of energy, write down the internal energy of a solid at room temperature. Show that the constant volume heat capacity of a solid is $C_v = 3Nk_B$. (This result is called the Dulong and Petit Law.)

   (b) Calculate the classical value of the molar specific heat of a solid in J K^{-1}, and also in calories K^{-1}. (1 calorie = 4.18 J)

5. A solid expands when its temperature is raised. A temperature increase of ΔT will produce an increase in the length of an object of $\Delta L = \alpha L_0 \Delta T$, where L_0 is the initial length and α is the *linear thermal expansion coefficient.*

(a) Calculate the contraction of a 30 cm aluminium ruler from room temperature (20°C) when put in liquid nitrogen ($T_{melt} = -196°\text{C}$) ($\alpha_{Al} = 23.1\ \mu\text{m}\cdot\text{m}^{-1}\cdot\text{K}^{-1}$).

(b) The Eiffel Tower is made of steel ($\alpha = 1.17 \times 10^{-5}\ \text{K}^{-1}$). At 20^o C it has a height of 320 m. Calculate how much shorter it is at -10^o C.

(b) The *volume thermal expansion coefficient*, β, is defined such that a temperature increase ΔT produces a volume increase $\Delta V = \beta V_0 \Delta T$, where V_0 is the initial volume. Show that for small temperature changes $\beta = 3\alpha$.

6. A system has *only* two microstates with energies E and $E + \Delta E$ ($\Delta E > 0$). Assuming that the system is in thermal equilibrium;

   (a) If there are N indistinguishable particles in total in either of the two states, calculate the populations of both levels.

   (b) Calculate the relative populations when

      i. $k_B T = 0.1\,\Delta E$
      ii. $k_B T = 0.3\,\Delta E$
      iii. $k_B T = \Delta E$
      iv. $k_B T = 10\,\Delta E$

   (c) The bond energy for a rubidium lattice is 0.023 eV per atom. Calculate the temperature at which the mean kinetic energy of a rubidium atom equals this energy. What would you expect to happen to rubidium at this temperature.

   (d) The first ionisation energy of rubidium is 4.2 eV. Calculate the temperature at which the kinetic energy of a rubidium atom is equal to this energy. What would you expect to happen to rubidium at this temperature.

   (e) Rubidium was the first substance to be made into a Bose-Einstein condensate. To make the condensate the de Broglie wavelength of the Rb atom must be greater than the mean spacing of the atoms $= n^{-1/3}$. Assuming that the mean kinetic energy is equivalent to $\frac{3}{2}k_B T$, calculate the mean wavelength of the condensate atoms λ. Hence show that,

   $$T_c < \frac{h^2}{3mk_B} n^{2/3}$$

   (f) Calculate the temperature required to create a Bose-Einstein condensate of 2000 Rb atoms ($A = 86$) trapped in a cube with sides of length 10 μm. What is the corresponding quantum of energy between the condensed and excited states?

7. Sound waves are compression waves. Regions of higher density (and thus pressure) exand, and exert a force on neighbouring regions (in a gas due to collisions), which causes it to compress and so the compression to propagate as a wave.

   (a) The speed of sound is defined as $c_s = \sqrt{\frac{\text{bulk modulus}}{\text{density}}}$, where the bulk modulus is defined as $K = -V\left(\frac{\partial P}{\partial V}\right)$

   Assuming sound waves in a gas can be described by an adiabatic equation of state, show that $c_s = \sqrt{\gamma P/\rho}$.

(b) Rewrite the sound speed in terms of temperature for an ideal gas. Hence calculate the sound speed in air ($\gamma = 1.4$), and in pure helium gas both at STP. (For air, the mean molecular mass $\bar{A} = 29.0$, and for helium $A = 4.0$).

8. Consider an exactly neutral hydrogen plasma in which $n_e = n_i = n_0$ (n_e and n_i are the electron and ion number densities) and the charge density $\rho_q = (n_i - n_e)\, e$ is zero everywhere. If all the electrons in a layer of width δ are shifted distance δ in one direction a charged layer will be formed in the plasma.

   (a) Show that this sets up an electric field with a the maximum value of $E_{max} = \dfrac{n_0 e d}{\epsilon_0}$. (Hint set up Gauss' law as for a capacitor).

   (b) The space charge field set-up tries to restore the displaced charge sheet of electrons back to their original position. The equation of motion for the electrons in the charge sheet can be written,
   $m_e \dfrac{\mathrm{d}\delta}{\mathrm{d}t} = -eE_{max}$
   Show that this implies SHM for the charge sheet, with a frequency equal to the plasma frequency. Why can the motion of the ions be ignored?

   (c) The work done to move the electrons is stored in the electric field. Given that the energy density in an electric field is $u_E = \frac{1}{2}\epsilon_0 E^2$ (in J m^{-3}), write down an expression for the maximum value of u_E.

   (d) Small non-neutral regions can arise spontaneously in the plasma as a result of thermal fluctuations. The energy to form them comes from the translational kinetic energy of the electrons. The average energy per unit volume available to form the charged layer described above is $n_0 \times \frac{1}{2}k_B T$ (the relevant electron motion corresponds to one degree of freedom). Equating this to the maximum u_E gives an expression for δ_{max}, the maximum width of the layer. Show that $\delta_{max} = \left(\dfrac{\epsilon_0 k_B T}{n_0 e^2}\right)^{1/2}$ (= the Debye length).

   (e) Calculate the plasma frequency and Debye length in a magnetically confined fusion plasma in which $n_0 = 10^{20}$ m^{-3} and $T = 10^8$ K.

Numerical Answers

1. (a) 89.8%, (b) $r \approx 10$ cm
2. (a) 4.8 ms^{-1}, (b) 8.29×10^4 s
3. (a) 1.57×10^{-10} m, (b) 128 K.
4. (b) 24.9 JK^{-1}mol^{-1}, 6.0 cal/K.
5. (a) 1.5 mm (b) 11.2 cm.
7. (b) 331 m/s, 973 m/s
8. (f) 5.6×10^{11} Hz, 6.91×10^{-5} m.