Problem Sheet 2
Lectures 5 and 6

Learning Outcomes

Jargon

Isothermal atmosphere, distribution function, velocity component distribution, Maxwell speed distribution, moment of a distribution function, velocity space.

Notation

$f(v_x)$, $f(v)$, v_{mp}, $< v >$, $< v^2 >$, v_{rms}.

Concepts

Differential equation for the variation of pressure with height in an isothermal atmosphere (derivation, and solving it to find the variation of number density with height); Boltzmann law; normalizing distribution function; deriving the expression for v_{mp} in a Maxwell distribution; given appropriate standard integrals, be able to find $< v >$ and $< v^2 >$.

Problems

1. The velocity component distribution function has the form $f(v_x) = A\mathrm{e}^{-\alpha v_x^2}$ where $\alpha = m/2k_BT$ and A is a constant. Use the fact that a molecule must have *some* value of v_x between $-\infty$ and $+\infty$ to show that $A = \left(\dfrac{m}{2\pi k_BT}\right)^{1/2}$.

2. Show that the most probable speed in a Maxwell distribution is $v_{mp} = \left(\dfrac{2k_BT}{m}\right)^{1/2}$.

3. In Sec. 5.5 of the lectures we found that the average speed in a Maxwell distribution is given by $< v >= 4\pi A^3 \int_0^\infty v^3 \mathrm{e}^{-\alpha v^2} \mathrm{d}v$. Show that $< v >= \left(\dfrac{8k_BT}{\pi m}\right)^{1/2}$.

4. The mean square speed, $< v^2 >$, i.e., the average value of v^2, is given by $\int_0^\infty v^2 f(v)\ \mathrm{d}v$ (integrals of the form $\int_0^\infty v^n f(v)\ \mathrm{d}v$ are called moments of the distribution function). Show that $< v^2 >= \dfrac{3k_BT}{m}$, and hence write down an expression for the average kinetic energy of a monatomic molecule in a Maxwell distribution.

5. We can identify three characteristic speeds in a Maxwell distribution: (1) v_{mp}, (2) $< v >$, (3) $v_{rms} = (< v^2 >)^{1/2}$, i.e., the root mean square speed. Calculate the values of these three speeds for O_2 molecules at 20^oC (the atomic mass of Oxygen is 16.0).

6. In Problem Sheet 1 (Q. 2) we considered the plasma in a fusion reactor in which the ions and electrons both have a temperature of 10^8K. The ions are a mixture of Deuterium and Tritium. For each of the three types of particles in the plasma (Deuterium ions, mass 3.34×10^{-27} kg, Tritium ions, mass 5.01×10^{-27} kg, and electrons, mass 9.11×10^{-31} kg) calculate
   (a) the average particle speed,
   (b) the average particle energy.

7. Show by integrating over the velocity component distribution function that $< v_x >= 0$.

8. In Classwork I we wrote the probability of a molecule being between heights z and $z + \mathrm{d}z$ in an isothermal atmosphere as $p(z)\mathrm{d}z$, where $p(z) = \frac{1}{\lambda}\mathrm{e}^{-z/\lambda}$, and $\lambda = \frac{k_B T}{mg}$. Show that the average height of a molecule in the atmosphere is λ, and hence obtain an expression for the average potential energy in terms of T. [Hint: this question involves an integral which is not on the list in Handout 1. You might consider doing it by parts.]

Numerical Answers

5. 390 ms^{-1}, 440 ms^{-1}, 478 ms^{-1}.

6. (a) Deuterium ions: 1.03×10^6 ms^{-1},
   Tritium ions: 8.37×10^5 ms^{-1},
   electrons: 6.21×10^7 ms^{-1}.
   (b) 2.07×10^{-15} J for all three types.