Vibrations & Waves Problem Sheet 4: Answers

1) a). $\dfrac{d\omega^2}{dk^2} = \dfrac{d\omega^2}{d\omega}\dfrac{d\omega}{dk}\dfrac{dk}{dk^2} = 2\omega \cdot v_g \left(\dfrac{dk^2}{dk}\right)^{-1} = 2\omega\, v_g \dfrac{1}{2k} = v_g \dfrac{\omega}{k} = v_g v_p$ ※

b). $v_p(\omega) = c/n(\omega) = c/\sqrt{1-\omega_p^2/\omega^2}$. But $v_p = \omega/k$

$\Rightarrow c/\sqrt{1-\omega_p^2/\omega^2} = \omega/k \Rightarrow c^2/(1-\omega_p^2/\omega^2) = \omega^2/k^2$

$\Rightarrow \omega^2 = c^2k^2 + \omega_p^2$ ※.

c) From a) $\dfrac{d\omega^2}{dk^2} = v_g v_p$. From disp rel. $\dfrac{d\omega^2}{dk^2} = c^2 \Rightarrow v_g v_p = c^2$

$\Rightarrow v_g = c^2/v_p = c\sqrt{1-\omega_p^2/\omega^2}$ ※.

d) For $\omega > \omega_p$: $v_p(\omega) > c$, but $v_g < c$. NB $v_p > c$ is not in conflict with relativity since information carried at group velocity, v_g.

2) $\psi(z,t) = f(z - v_p t) = f(y)$ with $y = z - v_p t$

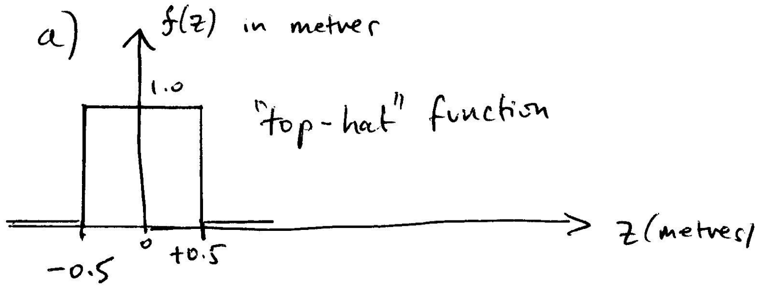
$$\frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial y}\frac{\partial y}{\partial t} = -v_p \frac{\partial \psi}{\partial y}, \quad \frac{\partial^2 \psi}{\partial t^2} = \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial t}\right)\frac{\partial y}{\partial t} =$$

$$-v_p \frac{\partial}{\partial y}\left(-v_p \frac{\partial \psi}{\partial y}\right) = v_p^2 \frac{\partial^2 \psi}{\partial y^2}$$

$$\frac{\partial \psi}{\partial z} = \frac{\partial \psi}{\partial y}\frac{\partial y}{\partial z} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial^2 \psi}{\partial z^2} = \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial z}\right)\frac{\partial y}{\partial z} = \frac{\partial^2 \psi}{\partial z^2}$$

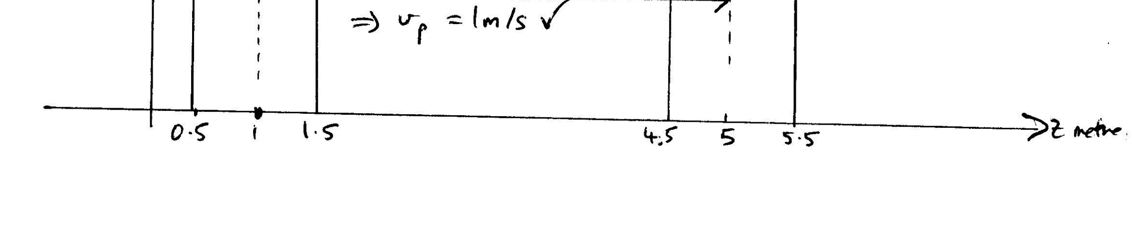
$$\Rightarrow \quad \frac{\partial^2 \psi}{\partial t^2} = v_p^2 \frac{\partial^2 \psi}{\partial z^2}$$

3) a)



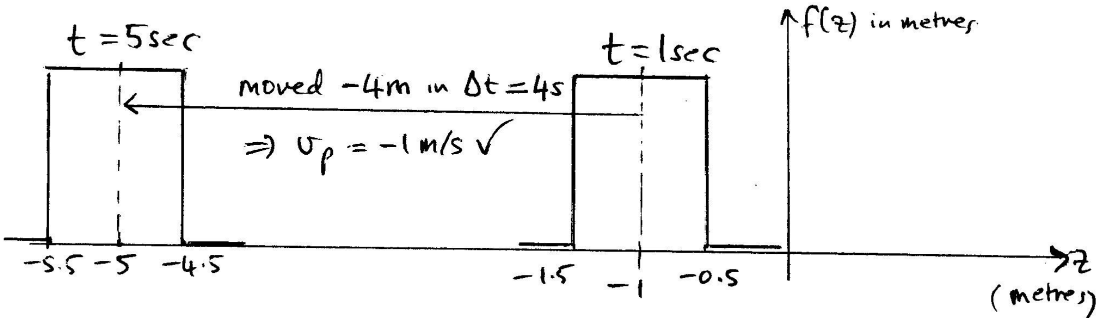
b) for $t = 1$sec, $v_p = 1$m/s, $f(z - v_p t) = f(z-1) = f(z)$ shifted in pos. z dir^n by 1m.

for $t = 5$sec, $v_p = 1$m/s, $f(z - v_p t) = f(z-5) = f(z)$ shifted in pos z dir^n by 5m.



c) for $t = 1\,\text{sec}$, $v_p = -1\,\text{m/s}$, $f(z - v_p t) = f(z+1) = f(z)$ shifted in neg. z dir$^{\underline{n}}$ by $1\,\text{m}$.

for $t = 5\,\text{sec}$, $v_p = -1\,\text{m/s}$, $f(z - v_p t) = f(z+5) = f(z)$ shifted in neg. z dir$^{\underline{n}}$ by $5\,\text{m}$.



4) a) $f(z) = A\exp(-z^2/w^2)$ "Gaussian" of width w.

$v_p = +5\,\text{m/s} \Rightarrow f(z - v_p t) = A\exp\left(-(z-5t)^2/w^2\right)$

b) $f(z) = (1 + z^2/w^2)^{-1}$ "Lorentzian" of width w.

$v_p = -2\,\text{m/s} \Rightarrow f(z - v_p t) = \left(1 + (z+2t)^2/w^2\right)^{-1}$

5) We had $f' = \left(\dfrac{v_p + u_R}{v_p - u_s}\right) f$ for $\xrightarrow{u_s} \quad \xleftarrow{u_R}$ (& change signs of u_R, u_s according to situation)

a) $u_R = v_T = 0$ $\quad \xleftarrow{u_s = u_B} \Rightarrow f'_T = \left(\dfrac{v_p + 0}{v_p - u}\right) f_B = \left(\dfrac{v_p}{v_p - u_B}\right) f_B$

[or from 1$^{\text{st}}$ principles, bat is moving towards tree $\Rightarrow \lambda$ emitted into air compressed by motion of source $\lambda'_T = \lambda_B - u_B T_B$ where $T_B = 1/f_B$

$f'_T = \dfrac{v_p}{\lambda'} = \dfrac{v_p}{\lambda_B - u_B T_B} = \dfrac{v_p}{v_p/f_B - u_B/f_B} = \left(\dfrac{v_p}{v_p - u_B}\right) f_B$.]

b) $\xleftarrow{u_R = u_m} \quad \xleftarrow{u_s = u_B} \Rightarrow f'_m = \left(\dfrac{v_p - u_R}{v_p - u_s}\right) f_B = \left(\dfrac{v_p - u_m}{v_p - u_B}\right) f_B$

{This is equivalent to $\xrightarrow{u_s = u_B} \quad \xrightarrow{u_R = u_m}$ }

[Or from 1$^{\text{st}}$ principles, since moth flying in same dir$^{\underline{n}}$ as bat it takes more time to receive each λ travelling in air $\Rightarrow$ period received T'_m stretched by motion $\lambda'_m = \lambda + u_m T_m$. $f'_m = \dfrac{v_p}{\lambda'_m} = \dfrac{v_p}{\lambda + u_m/f'_m}$

$\Rightarrow f'_m \lambda + u_m = v_p \Rightarrow f'_m v_p / f = v_p - u_m \Rightarrow f'_m = \left(\dfrac{v_p - u_m}{v_p}\right) f$

But f is freq. received by stationary receiver (ie f'_T)

$\Rightarrow f'_m = \left(\dfrac{v_p - u_m}{v_p - u_B}\right) f_B$.]

c) $u_S = u_T = 0$ $\quad \xleftarrow{u_R = u_B}$ Tree reflecting waves at freq. f_T'

$$\Rightarrow f_T'' = \left(\frac{v_p + u_R}{v_p - 0}\right) f_T' = \left(\frac{v_p + u_B}{v_p - u_B}\right) f_B$$

Or from 1st principles, since bat flying towards reflection from tree it takes less time to receive each wavelength from tree $\Rightarrow$ period received by bat T_T'' is compressed

$$\Rightarrow \lambda_T'' = \lambda_T' - u_B T_T''$$

$$f_T'' = \frac{v_p}{\lambda_T''} = \frac{v_p}{\lambda_T' - u_B/f_T''} \Rightarrow f_T''\lambda_T' - u_B = v_p$$

$$\Rightarrow f_T'' v_p / f_T' = v_p + u_B \Rightarrow f_T'' = \left(\frac{v_p + u_B}{v_p}\right) f_T' = \left(\frac{v_p + u_B}{v_p - u_B}\right) f_B$$

d) $\xleftarrow{u_S = u_M}$ $\quad \xleftarrow{u_R = u_B}$ Moth reflecting waves at freq f_M'

$$\Rightarrow f_M'' = \left(\frac{v_p + u_R}{v_p + u_S}\right) f_M' = \left(\frac{v_p + u_B}{v_p + u_M}\right)\left(\frac{v_p - u_M}{v_p - u_B}\right) f_B$$

Or from first principles, since moth moving away from bat, reflected waves from moth will have wavelength stretched by motion of source (moth)

$$\Rightarrow \lambda_{M.E} = \lambda_M' + u_M T_M'$$

where $\lambda_{M.E}$ is Doppler Shifted wavelength emitted by moth.

$$f_{ME} = \frac{v_p}{\lambda_{M.E}} = \frac{v_p}{\lambda_M' + u_M/f_M'}$$

$$= \frac{v_p}{\frac{v_p}{f_M'} + \frac{u_M}{f_M'}} = \left(\frac{v_p}{v_p + u_M}\right) f_M' = \left(\frac{v_p}{v_p + u_M}\right)\left(\frac{v_p - u_M}{v_p - u_B}\right) f_B$$

Bat is moving towards moth $\Rightarrow$ waves received by bat will be compressed by motion of receiver (bat)

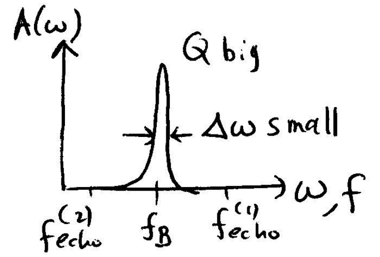
$$\lambda_M'' = \lambda_{M.E} - u_B T_M''$$

$$f_M'' = \frac{v_p}{\lambda_M''} = \frac{v_p}{\lambda_{ME} - u_B/f_M''} = \frac{v_p}{\frac{v_p}{f_{ME}} - \frac{u_B}{f_M''}} \Rightarrow v_p \frac{f_M''}{f_{ME}} - u_B = v_p$$

$$\Rightarrow f_M'' = \left(\frac{v_p + u_B}{v_p}\right) f_{ME} = \left(\frac{v_p + u_B}{\cancel{v_p}}\right)\left(\frac{\cancel{v_p}}{v_p + u_M}\right)\left(\frac{v_p - u_M}{v_p - u_B}\right) f_B$$

e) A high Q $\Rightarrow$ narrow bandwidth $\Delta\omega$ for bat's hearing

ie



Bat's ear is like a forced, damped SHM system (as is our own).

So a bat with high Q ears only hear well over narrow range of frequencies centred on their emission frequency.

The problem is that Doppler-Shifted echos (eg $f_{echo}^{(1),(2)}$) may lie outside their hearing range!

f) Want echos to be at 10kHz = frequency that bats hear best at in this example.

Using $u_B = 4\,m/s$, $u_M = 0.1\,m/s$, $v_p = 344\,m/s$

Tree: $$10\,kHz = f_T'' = \left(\frac{v_p + u_B}{v_p - u_B}\right) f_B = \left(\frac{344\,m/s + 4\,m/s}{344\,m/s - 4\,m/s}\right) f_B$$

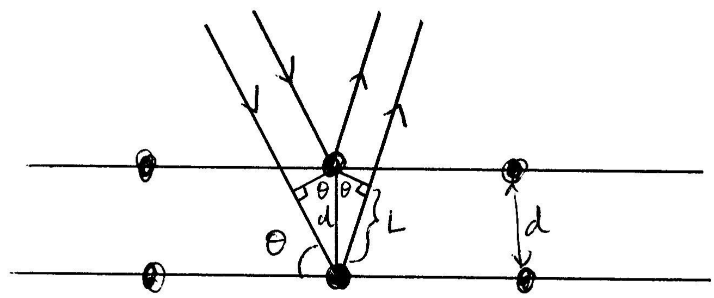
$$\Rightarrow f_B = 9.77\,kHz$$

Moth: $$10\,kHz = f_M'' = \left(\frac{v_p + u_B}{v_p + u_M}\right)\left(\frac{v_p - u_M}{v_p - u_B}\right) f_B = \left(\frac{344\,m/s + 4\,m/s}{344\,m/s + 0.1\,m/s}\right)\left(\frac{344\,m/s - 0.1\,m/s}{344\,m/s - 4\,m/s}\right) \times f_B$$

$$\Rightarrow f_B = 9.78\,kHz$$

So bat needs to sweep its emission frequency from 9.77 kHz to 9.78 kHz for echos from tree and moth to be at 10kHz where it hears best.

6) a)



$$L = d\sin\theta$$

path difference $\Delta r = 2L = 2d\sin\theta$

b) For constructive interference

$$\Delta r = n\lambda \qquad n = 0, \pm 1, \pm 2, \dots$$

$$\Rightarrow \quad 2d\sin\theta = n\lambda$$

$$\Rightarrow \quad \theta = \arcsin\left(\frac{n\lambda}{2d}\right) \qquad n = 0, \pm 1, \pm 2, \dots$$

c) x-rays (or electron waves $\lambda_e = \frac{h}{p_0}$) have comparable wavelengths to lattice spacing d $\Rightarrow$ will be diffracted.

7) <u>Past Exam Question</u>

(i)(a) $y_+(x,t) = A\cos(\omega t - kx + \varphi)$
$y_-(x,t) = -A\cos(\omega t + kx + \varphi)$

} here x is used as propagation dir^n instead of z - makes no difference (& y instead of ψ for displacement)

$$y_{total}(x,t) = y_+ + y_-$$
$$= A\left[\cos(\omega t - kx + \varphi) - \cos(\omega t + kx + \varphi)\right]$$
$$= -2A\sin(\omega t + \varphi)\sin(-kx)$$
$$= 2A\sin(kx)\sin(\omega t + \varphi) \longrightarrow \text{Standing Wave}$$

no $-k.x$ here

Boundary conditions:

$y_{total}(0,t) = 0 \Rightarrow \sin(kx) = 0$

$y_{total}(L - md, t) = 0 \Rightarrow \sin[k(L-md)] = 0$

$\Rightarrow k(L - md) = n\pi \Rightarrow \lambda_n = \frac{2(L-md)}{n}$

$n = 0, 1, 2, \dots$

(b) $v_p = \sqrt{\frac{T^*}{\sigma}}$ [nb: I used μ instead of σ for mass per unit length. T^* for the tension is to avoid confusion with period T. In most cases it's obvious from context.]

$$f_{n=1,m=0} = \frac{v_p}{\lambda_{n=1,m=0}} = \frac{v_p}{2L} \Rightarrow v_p = 2Lf = \sqrt{\frac{T^*}{\sigma}}$$

$$\Rightarrow T^* = 4L^2f^2\sigma = 4 \times (1\,\mathrm{m})^2 \times (10^3\,\mathrm{Hz})^2 \times 10^{-3}\,\mathrm{kg\,m^{-1}}$$
$$= 4000\,\mathrm{N}.$$

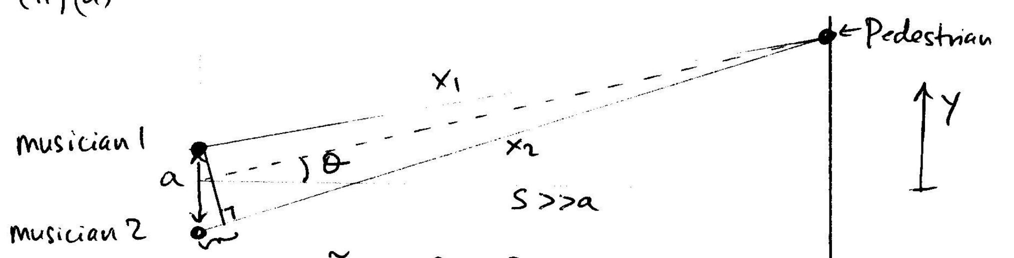
(c) $f_{beat} = |f_{n=1,m=0} - f_{n=1,m=2}|$, $L = 1\,\mathrm{m}$, $d = 0.05\,\mathrm{m}$, $T^* = 4000\,\mathrm{N}$, $\sigma = 10^{-3}\,\mathrm{kg\,m^{-1}}$

$$f_{n=1,m=0} = \frac{v_p}{\lambda_{n=1,m=0}} = \frac{v_p}{2L} = \sqrt{\frac{T^*}{\sigma}}\Big/(2L) = 1\,\mathrm{kHz}$$

$$f_{n=1,m=2} = \frac{v_p}{\lambda_{n=1,m=2}} = \frac{v_p}{2(L-2d)} = \sqrt{\frac{T^*}{\sigma}}\Big/2(L-2d) = 1.111\,\mathrm{kHz}$$

$$\Rightarrow f_{beat} = 111\,\mathrm{Hz}$$

(ii)(a)



$x_1 - x_2 \cong a\sin\theta$, θ small

NB different Symbols used compared to my lecture notes, but physics same!

Sound waves from musicians 1 & 2 will arrive in-phase at pedestrian ("constructive interference") when $x_1 - x_2 = a\sin\theta = p\lambda$ $p = 0, \pm 1, \pm 2, \ldots$

$$\Rightarrow \sin\theta_{max} = \frac{p\lambda}{a} = \frac{pV}{a\nu}$$ where V = velocity, ν = frequency of sound waves

[note: I use v_p & f for these quantities]

(b) $S >> a \Rightarrow \sin\theta_{max} \approx \theta_{max} = \frac{p v}{a \nu}$

$$Y/s = \tan\theta \approx \theta$$

For constructive interference (max sound intensity)

$$Y_{max} = \frac{s p v}{a \nu}$$

Distance between points of constructive interference (p changes by 1)

$$\Delta Y = \frac{s v}{a \nu} = \frac{50\,m \times 344\,m/s}{5\,m \times 10^3\,Hz} = 3.44\,m$$

(c) Intensity $I_0 = \frac{P_{av}}{A}$

A = area of hemisphere $= 2\pi r^2$

At interference maximum, $I_{max} = 4 I_0$

$$= \frac{4 P_{av}}{2\pi r^2} = \frac{2 P_{av}}{\pi r^2}$$

$$\Rightarrow I_{max} = \frac{2 \times 1\,W}{\pi \times (50\,m)^2} = 2.55 \times 10^{-4}\ W/m^2$$